On the Central Limit Theorem for Toeplitz Quadratic Forms of Stationary Sequences*

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Abstract

Let $X(t)$, $t = 0, \pm 1, \dots$, be a real-valued stationary Gaussian sequence with spectral density function $f(\lambda)$. The paper considers a question of applicability of central limit theorem (CLT) for Toeplitz type quadratic form Q_n in variables $X(t)$, generated by an integrable even function $g(\lambda)$. Assuming that $f(\lambda)$ and $g(\lambda)$ are regularly varying at $\lambda = 0$ of orders α and β respectively, we prove CLT for standard normalized quadratic form Q_n in the critical case $\alpha + \beta = 1/2$.
We also show that CLT is not valid under the single condition that the asymptotic variance of Q_n is separated from zero and infinity.

Key words and phrases. Stationary Gaussian sequence, spectral density, Toeplitz type quadratic forms, central limit theorem, asymptotic variance, regularly varying functions.

AMS 2000 subject classifications. Primary 60G10, 60F05; Secondary 60G15

1 Introduction

Let $X(t)$, $t = 0, \pm 1, ...$ be a centered ($\mathbb{E}X(t) = 0$) real-valued stationary Gaussian sequence with spectral density $f(\lambda)$ and covariance function $r(t)$, i. e.

$$X(t) = \int_{-\pi}^{\pi} e^{i\lambda t} f(\lambda)\, d\lambda. \tag{1.1}$$

We consider a question concerning asymptotic distribution (as $n \to \infty$) of the following Toeplitz type quadratic forms of the process $X(t)$:

$$Q_n = \sum_{k,j=1}^{n} a(k-j) X(k) X(j), \tag{1.2}$$

*This work was completed with the support of ANSEF Grant No. PS58 and NFSAT/CRDF Grant No. MA 070-02/12011.

where

$$a(k) = \int_{-\pi}^{\pi} e^{i\lambda k}\, g(\lambda)\, d\lambda, \;\; k = 0, \pm 1, ... \tag{1.3}$$

are the Fourier coefficients of some real, even, integrable function $g(\lambda)$ on $\mathbb{T} = [-\pi, \pi]$. We will refer $g(\lambda)$ as a generating function for the quadratic form Q_n. Throughout the paper the functions $f(\lambda)$ and $g(\lambda)$ are assumed to be 2π-periodic.

The limit distribution of the random variables (1.2) is completely determined by the spectral density $f(\lambda)$ and the generating function $g(\lambda)$, and depending on their properties it can be either Gaussian (that is, Q_n with an appropriate normalization obey central limit theorem), or non-Gaussian. We naturally arise the following two questions:

a) Under what conditions on $f(\lambda)$ and $g(\lambda)$ will the limits be Gaussian?

b) Describe the limit distributions, if they are non-Gaussian.

In this paper we essentially discuss the question a). This question goes back to the classical monograph by Grenander and Szegö [9], where they considered this problem, as an application of their theory of the asymptotic behavior of the trace of products of truncated Toeplitz matrices.

Later this problem was studied by I. Ibragimov [11] and M. Rosenblatt [12], in connection with statistical estimation of the spectral ($F(\lambda)$) and covariance ($r(t)$) functions, respectively. Since 1986, there has been a renewed interest in questions a) and b), related to the statistical inferences for long-range dependent processes (see, e.g., Avram [1], Fox and Taqqu [4], Giraitis and Surgailis [8], Terrin and Taqqu [14], Taniguchi [17], Taniguchi and Kakizawa [18], and references therein).

Avram [1], Fox and Taqqu [4] and Giraitis and Surgailis [8] have obtained sufficient conditions for quadratic form Q_n to obey the central limit theorem (CLT). Below we use the following notation:
By $\widetilde{Q}_n$ we denote the normalized quadratic form:

$$\widetilde{Q}_n = \frac{1}{\sqrt{n}}\,(Q_n - EQ_n) \tag{1.4}$$

The notation

$$\widetilde{Q}_n \Longleftarrow\!\!\!\Longrightarrow N(0, \sigma^2) \tag{1.5}$$

will mean that the distribution of the random variable $\widetilde{Q}_n$ tends (as $n \to \infty$) to the centered normal distribution with variance σ^2.

By $T_n(f)$ and $T_n(g)$ we denote the $n \times n$ Toeplitz matrices generated by functions f and g, respectively, i.e.

$$T_n(f) = ||r(k-j)||_{k,j=\overline{1,n}} \quad and \quad T_n(g) = ||a(k-j)||_{k,j=\overline{1,n}}, \tag{1.6}$$

where $r(k)$ and $a(k)$ are as in (1.1) and (1.3), respectively. By C, M, C_k, M_k we will denote constants that can vary from line to line.

Theorem A (Avram). *Let the spectral density $f(\lambda)$ and the generating function $g(\lambda)$ be such that $f(\lambda) \in L^{p_1}(\mathbb{T})$, $g(\lambda) \in L^{p_2}(\mathbb{T})$, where $p_1, p_2 \geq 1$ and $1/p_1 + 1/p_2 \leq 1/2$. Then (1.5) holds with σ^2 given by*

$$\sigma^2 = 16\pi^3 \int_{-\pi}^{\pi} f^2(\lambda) g^2(\lambda)\, d\lambda. \tag{1.7}$$

Remark 1.1. For $p_1 = p_2 = \infty$ Theorem A was first established by Grenander and Szegö ([9], theorem 11.6), while the case $p_1 = 2$, $p_2 = \infty$ was proved by Ibragimov [11] and Rosenblatt [12].

Theorem B (Fox and Taqqu)**.** *Assume that the conditions hold:*
a) the discontinuities of $f(\lambda)$ and $g(\lambda)$ have Lebesgue measure zero, and $f(\lambda)$ and $g(\lambda)$ are bounded on $[\delta, \pi]$ for all $\delta > 0$;
b) there exist $\alpha < 1$ and $\beta < 1$ such that $\alpha + \beta < \frac{1}{2}$,

$$f(\lambda) \sim |\lambda|^{-\alpha} L_1(\lambda) \quad \text{as} \quad \lambda \to 0 \tag{1.8}$$

and

$$g(\lambda) \sim |\lambda|^{-\beta} L_2(\lambda) \quad \text{as} \quad \lambda \to 0, \tag{1.9}$$

where $L_1(\lambda)$ and $L_2(\lambda)$ are slowly varying at $\lambda = 0$ functions. Then (1.5) holds with σ^2 as in (1.7).

The proofs of Theorems A and B in [1] and [4] are based on the well–known representation of the k–th order cumulant $\chi_k(\cdot)$ of $\widetilde{Q}_n$ (see, e.g. [9], [11]):

$$\chi_k(\widetilde{Q}_n) = \begin{cases} 0, & \text{for } k = 1 \\ n^{-k/2} 2^{k-1} (k-1)! \ \mathrm{tr} \, [T_n(f) T_n(g)]^k, & \text{for } k \geq 2, \end{cases}$$

where $\mathrm{tr}[A]$ stands for the trace of a matrix A.

A different approach [8] extended Theorem A to linear sequences. In the Gaussian case their result can be formulated as follows.

Theorem C (Giraitis and Surgailis)**.** *Assume that*

$$\chi_2(\widetilde{Q}_n) = \frac{2}{n} \ tr\big[T_n(f) T_n(g)\big]^2 \longrightarrow 16\pi^3 \int_{-\pi}^{\pi} f^2(\lambda) g^2(\lambda) \, d\lambda < \infty. \tag{1.10}$$

Then (1.5) holds with σ^2 as in (1.7).

In [1] and [4] (see, also, [8]) was established that each of the conditions of Theorems A and B imply (1.10), i. e. (1.10) is weaker than the conditions of Theorems A and B. Unfortunately (1.10) is not an explicit condition. In [8] also was obtained the following explicit sufficient condition.

Theorem D (Giraitis and Surgailis)**.** *Let $f \in L^2(\mathbb{T})$, $g \in L^2(\mathbb{T})$, $fg \in L^2(\mathbb{T})$ and*

$$\int_{-\pi}^{\pi} f^2(\lambda) g^2(\lambda - \mu) \, d\lambda \longrightarrow \int_{-\pi}^{\pi} f^2(\lambda) g^2(\lambda) \, d\lambda \quad \text{as} \quad \mu \to 0. \tag{1.11}$$

Then (1.5) holds with σ^2 as in (1.7).

In the same paper [8] Giraitis and Surgailis conjectured that (1.10) holds under the single condition that the integral on the right hand side of (1.10) is finite.

In [6] one of the authors answered this conjecture negatively. We recall this result. Consider the functions

$$f_0(\lambda) = \begin{cases} \left(\frac{2^s}{s^2}\right)^{1/p}, & \text{if } 2^{-s-1} \leq \lambda \leq 2^{-s}, \ s = 2m \\ 0, & \text{if } 2^{-s-1} \leq \lambda \leq 2^{-s}, \ s = 2m+1 \end{cases} \tag{1.12}$$

and

$$g_0(\lambda) = \begin{cases} \left(\frac{2^s}{s^2}\right)^{1/q}, & \text{if } 2^{-s-1} \le \lambda \le 2^{-s},\ s = 2m+1 \\ \\ 0, & \text{if } 2^{-s-1} \le \lambda \le 2^{-s},\ s = 2m, \end{cases} \quad , \tag{1.13}$$

where m is a positive integer and $p, q \ge 1$.
It is easy to see that $f_0(\lambda) \in L^p(\mathbb{T})$, $g_0(\lambda) \in L^q(\mathbb{T})$, $f_0(\lambda)\, g_0(\lambda) \in L^r(\mathbb{T})$ for every r and

$$\sigma^2 = 16\pi^3 \int_{-\pi}^{\pi} f_0^2(\lambda) g_0^2(\lambda)\, d\lambda = 0.$$

On the other hand, in [6] was proved that for $\frac{1}{p} + \frac{1}{q} > 1$

$$\chi_2(\widetilde{Q}_n) = \frac{2}{n}\, \mathrm{tr}\, (T_n(f_0)T_n(g_0))^2 \longrightarrow \infty \quad \text{as} \quad n \to \infty, \tag{1.14}$$

and thereby the convergence in (1.10) breaks down.
In [6] was conjectured, that the condition

$$0 < \int_{-\pi}^{\pi} f^2(\lambda)\, g^2(\lambda)\, d\lambda < \infty$$

implies the convergence in (1.10).

The problem b), i.e. description of the limit distributions of quadratic forms Q_n, if they are non-Gaussian was considered by Terrin and Taqqu in [14], [15]. Let $f(\lambda) = |\lambda|^{-\alpha} L_1(\lambda)$ and $g(\lambda) = |\lambda|^{-\beta} L_2(\lambda)$, where $L_1(\lambda)$ and $L_2(\lambda)$ are slowly varying at 0, and are bounded on bounded intervals. In [14], [15] was proved that if $\alpha < 1$, $\beta < 1$, and $\alpha + \beta > 1/2$ then the random variable

$$\widehat{Q}_n = \frac{1}{n^{\alpha+\beta} L_1(1/n) L_2(1/n)}\, (Q_n - EQ_n) \tag{1.15}$$

converges in distribution to some non-Gaussian random variable $Y(\alpha, \beta)$, which can be represented as a double Wiener-Itô integral.

Note that the slowly varying functions $L_1(\lambda)$ and $L_2(\lambda)$ are of importance because they provide a great flexibility in the choice of functions $f(\lambda)$ and $g(\lambda)$. In [14] was proved that they influence only the normalization in (1.15) and not the limit $Y(\alpha, \beta)$. In this paper we prove that in the critical case $\alpha + \beta = 1/2$ the limit distribution of the standard normalized quadratic form Q_n depends on functions $L_1(\lambda)$ and $L_2(\lambda)$.

The critical case $\alpha + \beta = 1/2$ was partially investigated by Taqqu and Terrin in [16]. Starting from $Y(\alpha, \beta)$, which exists only when $\alpha + \beta > 1/2$, they showed that when $0 < \alpha < 1$, $0 < \beta < 1$ the random variable $(\alpha + \beta - 1/2)Y(\alpha, \beta)$ converges in distribution to a Gaussian random variable as $\alpha + \beta$ approaches to $1/2$.

Assuming that $f(\lambda)$ and $g(\lambda)$ are regularly varying at $\lambda = 0$ of orders α and β respectively, we prove CLT for standard normalized quadratic form Q_n in the critical case $\alpha + \beta = 1/2$. We also show that CLT for Q_n is not valid under the single condition that the asymptotic variance of Q_n is separated from zero and infinity.

2 Results

Let SV be the class of slowly varying at zero functions $u(\lambda)$ satisfying

$$u(\lambda) \in L^{\infty}(\mathbb{R}), \quad \lim_{\lambda \to 0} u(\lambda) = 0, \quad u(\lambda) = u(-\lambda),\ 0 < u(\lambda) < u(\mu) \text{ for } 0 < \lambda < \mu.$$

Theorem 2.1. *Let*

$$f(\lambda) \le |\lambda|^{-\alpha} L_1(\lambda) \tag{2.1}$$

and

$$|g(\lambda)| \le |\lambda|^{-\beta} L_2(\lambda), \tag{2.2}$$

where

$$\alpha < 1,\ \beta < 1,\ \alpha + \beta \le 1/2 \quad \text{and} \quad L_i \in SV,\ \lambda^{\alpha+\beta} L_i \in L^2(\mathbb{T}),\ i = 1, 2. \tag{2.3}$$

Then (1.5) holds with σ^2 as in (1.7).

Remark 2.1. Examples of spectral density $f(\lambda)$ and generating function $g(\lambda)$ satisfying Theorem 2.1 provide the functions

$$f(\lambda) = |\lambda|^{-\alpha} |\ln|\lambda||^{-\gamma} \quad \text{and} \quad g(\lambda) = |\lambda|^{-\beta} |\ln|\lambda||^{-\gamma},$$

where $\alpha < 1$, $\beta < 1$, $\alpha + \beta \le 1/2$ and $\gamma > 1/2$.

For $f, g \in L^1(\mathbb{T})$ we denote

$$\varphi(t_1, t_2, t_3) = \int_{-\pi}^{\pi} f(u) g(u - t_1) f(u - t_2) g(u - t_3)\, du. \tag{2.4}$$

Theorem 2.2. *If the function $\varphi(t_1, t_2, t_3) \in L^2(\mathbb{T}^3)$ is continuous at $(0, 0, 0)$, then (1.5) holds with σ^2 as in (1.7).*

Proposition 2.1. *Theorem 2.2 implies Theorems A and D.*

Remark 2.2. For functions $f(\lambda) = \lambda^{-3/4}$ and $g(\lambda) = \lambda^{3/4}$ satisfying conditions of Theorem B the function $\varphi(t_1, t_2, t_3)$ is not defined for $t_2 = 0$, $t_1 \ne 0$, $t_3 \ne 0$. This shows that Theorem 2.2 generally does not imply Theorem B.

The next result shows that the condition of positiveness and finiteness of asymptotic variance of quadratic form Q_n is not sufficient for Q_n to obey CLT.

Proposition 2.2. *There exist spectral density $f(\lambda)$ and generating function $g(\lambda)$, such that*

$$0 < \int_{-\pi}^{\pi} f^2(\lambda)\, g^2(\lambda)\, d\lambda < \infty \tag{2.5}$$

and

$$\lim_{n\to\infty} \sup \chi_2(\widetilde{Q}_n) = \lim_{n\to\infty} \sup \frac{2}{n}\ tr\,(T_n(f) T_n(g))^2 = \infty, \tag{2.6}$$

that is, the condition (2.5) does not guarantee convergence in (1.10).

3 Preliminaries

Recall (see [3], [13]) that a positive function $u(x)$ is called slowly varying at zero, if

$$\lim_{x\to 0} \frac{u(\lambda x)}{u(x)} = 1,$$

for any $\lambda > 0$. We list some properties of slowly varying functions. The following property is well known (see, e.g., [13]).

Lemma 3.1. *Let $u(x), v(x),\ x \in \mathbb{R}$ be slowly varying at zero functions. Then*

a) For any $p < 1$

$$\int_o^y x^{-p} u(x)dx = O\left(y^{1-p}u(y)\right) \quad \text{as} \quad y \to 0.$$

b) The function $x^p u(x)$ is increasing in some interval $(0, \delta)$, if $p > 0$ and is decreasing, if $p < 0$.

c) The functions uv and $\frac{u}{v}$ are slowly varying at zero functions.

Lemma 3.2. *Given functions $u, v \in SV$ and numbers $p, q < 1$, $p + q > 1$, there exists a constant $M > 0$ such that*

$$\int_{\mathbb{T}} |x|^{-p}|x-y|^{-q}u(x)v^{-1}(x-y)dx \leq M|y|^{1-p-q}u(y)v^{-1}(y), \quad y \in \mathbb{T}. \tag{3.1}$$

Proof. Denote $Q(x, y) = |x|^{-p}|x-y|^{-q}u(x)v^{-1}(x-y)$. It is not hard to check that for any $\delta > 0$

$$\sup_{|y|>\delta} \int_{\mathbb{T}} Q(x, y)dx < \infty \quad \text{and} \quad \min_{|y|>\delta} y^{1-p-q}u(y)v^{-1}(y) > 0.$$

Therefore it is enough to prove (3.1) for $y \in (-\delta, \delta)$ with sufficiently small $\delta > 0$. Applying Lemma 3.1 a) we get

$$\begin{aligned} \int\limits_{0<|x|<|y|/2} Q(x, y)dx &\leq \left(\frac{|y|}{2}\right)^{-q} v^{-1}\left(\frac{y}{2}\right) \int\limits_{0<|x|<|y|/2} |x|^{-p}u(x)dx \\ &\leq Cy^{1-p-q}u(y)v(y), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \int\limits_{|y|/2<|x|<2|y|} Q(x, y)dx &\leq \left(\frac{|y|}{2}\right)^{-p} u(2|y|) \int\limits_{|y|/2<|x|<2|y|} |x-y|^{-q}v^{-1}(x-y)dx \\ &\leq C|y|^{-p}u(|y|) \int\limits_{0<|x|<4|y|} |x|^{-q}v^{-1}(x)dx \leq Cy^{1-p-q}u(y)v(y), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \int\limits_{2|y|<|x|<\pi} Q(x, y)dx &\leq |y|^{-p}v^{-1}(y) \int\limits_{2|y|<|x|<\pi} |x|^{-q}u(x)dx \\ &\leq Cy^{1-p-q}u(y)v(y). \end{aligned} \tag{3.4}$$

From (3.2)-(3.4) we obtain (3.1). Lemma 3.2 is proved. □

The proof of the next lemma is similar.

Lemma 3.3. *Given functions $u, w \in SV$ satisfying $\int_{\mathbb{T}} x^{-1}u(x)w^{-3}(x)dx < \infty$. For any $q \in (0,1)$ there exists a constant $M > 0$ such that*

$$\int_{\mathbb{T}} |x|^{-1}|x-y|^{-q}u(x)w^{-2}(x)w^{-1}(x-y)dx \le M|y|^{-q}w^{-3}(y), \quad y \in \mathbb{T}.$$

We denote by $D_n(x)$ the Dirichlet kernel:

$$D_n(x) = \frac{\sin(nx/2)}{\sin(x/2)}. \tag{3.5}$$

It is not hard to see that

$$|D_n(x)| \le \min\{n, |x|^{-1}\}, \quad |D_n(x)| \le Cn\psi_n(x), \quad x \in \mathbb{T} \tag{3.6}$$

where

$$\psi_n(x) = (1 + n|x|)^{-1}.$$

Lemma 3.4. *For any function $w \in SV$ and a number $t \in (0,1)$ there exists a constant $M > 0$ such that*

$$|D_n(x)| \le Mw\,(1/n)\,n^t|x|^{t-1}w^{-1}(x).$$

Proof. According to Lemma 3.1 b) the functions $x^{t-1}w^{-1}(x)$ and $x^{-t}w(x)$ are decreasing in some interval $(0,\delta)$. Since

$$\min\{w\,(1/n)\,n^t|x|^{t-1}w^{-1}(x)\} > 0,$$

we can assume that $n^{-1} < \delta$ and $|x| < \delta$. Now, if $|x| \le n^{-1}$ then $n^{1-t}w^{-1}(1/n) \le x^{t-1}w^{-1}(x)$ and (3.6) implies

$$|D_n(x)| \le n = w\,(1/n)\,n^t n^{1-t}w^{-1}\,(1/n) \le w\,(1/n)\,n^t|x|^{t-1}w^{-1}(x).$$

The proof in the case $|x| > n^{-1}$ is similar. Lemma 3.4 is proved. □

The following lemma was proved in [8].

Lemma 3.5. *For any $\delta \in (0,1)$ there exists a constant $C_\delta > 0$ such that*

$$n\int_{\mathbb{T}} \psi_n(x-y)\psi_n(x-z)dx \le C_\delta \psi_n^{1-\delta}(y-z), \quad y, z \in \mathbb{T}.$$

Denote

$$\Phi_n(x_1,x_2,x_3) = \frac{1}{(2\pi)^3 n}D_n(x_1)D_n(x_2)D_n(x_3)D_n(x_1+x_2+x_3), \tag{3.7}$$

where $D_n(x)$ is as in (3.5). Given $\alpha \in (0,\pi)$ we set

$$\mathbb{E}_\alpha = \{|x| \le \alpha\} = \{(x_1,x_2,x_3);\, |\mathbf{x}_k| \le \alpha,\, k = 1,2,3\},$$
$$\mathbb{E}_\alpha^c = \{|x| \le \pi\} \setminus \{|x| \le \alpha\}.$$

Lemma 3.6. *The kernel $\Phi_n(\mathbf{x})$ defined by (3.7) with $\mathbf{x} = (x_1, x_2, x_3)$ possesses the following properties:*

a) $\displaystyle\int_{\mathbb{T}^3} \Phi_n(\mathbf{x})\, d\mathbf{x} = 1;$

b) $\displaystyle\sup_n \int_{\mathbb{T}^3} |\Phi_n(\mathbf{x})|\, d\mathbf{x} = C_1 < \infty;$

c) for any ε $(0 < \varepsilon \le \pi)$

$$\lim_{n\to\infty} \int_{\mathbb{E}^c_\varepsilon} |\Phi_n(\mathbf{x})|\, d\mathbf{x} = 0,$$

d) for any $\delta > 0$ there exists a positive constant M_δ such that

$$\int_{\mathbb{E}^c_\delta} \Phi^2_n(\mathbf{x}) d\mathbf{x} \le M_\delta \quad \textit{for} \quad n = 1, 2, \dots. \tag{3.8}$$

Proof. Proofs of a) - c) can be found in [2] (Lemma 3.1). To prove d) first observe that

$$\int_{\mathbb{T}} D^2_n(x) dx \le C\, n \quad \text{and} \quad |D_n(x)| \le C_\delta \quad \text{for} \quad |x| > \delta,\ n = 1, 2, \dots, \tag{3.9}$$

where $D_n(x)$ is the Dirichlet kernel, while C and C_δ are some positive constants. We have

$$\begin{aligned} \int_{\mathbb{E}^c_\delta} \Phi^2_n(\mathbf{x}) d\mathbf{x} \ &\le \int\limits_{|x_1|>\delta} \Phi^2_n(\mathbf{x}) d\mathbf{x} + \int\limits_{|x_2|>\delta} \Phi^2_n(\mathbf{x}) d\mathbf{x} \\ &+ \int\limits_{|x_3|>\delta} \Phi^2_n(\mathbf{x}) d\mathbf{x} =: I_1 + I_2 + I_3. \end{aligned} \tag{3.10}$$

Clearly, it is enough to estimate I_1. We have

$$\begin{aligned} I_1 \ &\le \int\limits_{|x_1|>\delta,\ |x_2|>\delta/3} \Phi^2_n(\mathbf{x}) d\mathbf{x} + \int\limits_{|x_1|>\delta,\ |x_3|>\delta/3} \Phi^2_n(\mathbf{x}) d\mathbf{x} \\ &+ \int\limits_{|x_1|>\delta,\ |x_2|\le\delta/3,\ |x_3|\le\delta/3} \Phi^2_n(\mathbf{x}) d\mathbf{x} =: I^{(1)}_1 + I^{(2)}_1 + I^{(3)}_1. \end{aligned} \tag{3.11}$$

Using (3.9) we obtain

$$I^{(1)}_1 \le C_\delta \cdot \frac{1}{n^2} \int_{\mathbb{T}^3} D^2_n(x_3) D^2_n(x_1 + x_2 + x_3) dx_1 dx_2 dx_3 \le M_\delta. \tag{3.12}$$

Likewise,

$$I^{(2)}_1 \le M_\delta. \tag{3.13}$$

Now, observing that in the integral $I^{(3)}_1$, $|x_1 + x_2 + x_3| > \delta/3$, from (3.9) we find

$$I^{(3)}_1 \le C_\delta \cdot \frac{1}{n^2} \int_{\mathbb{T}^3} D^2_n(x_2) D^2_n(x_3) dx_1 dx_2 dx_3 \le M_\delta. \tag{3.14}$$

From (3.12) – (3.14) we obtain (3.11). Lemma 3.6 is proved. □

Lemma 3.7. *Let the function* $\Psi(\mathbf{u}) \in L^2(\mathbb{T}^3)$ *be continuous at* $\mathbf{u} = (0,0,0)$*. Then*

$$\lim_{n\to\infty} \int_{\mathbb{T}^3} \Psi(\mathbf{u})\Phi_n(\mathbf{u})d\mathbf{u} = \Psi(0,0,0), \tag{3.15}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\Phi_n(\mathbf{u})$ *is defined by (3.7).*

Proof. By Lemma 3.6 a) we have

$$R_n := \int_{\mathbb{T}^3} \Psi(\mathbf{u})\Phi_n(\mathbf{u})d\mathbf{u} - \Psi(0,0,0) = \int_{\mathbb{T}^3} [\Psi(\mathbf{u}) - \Psi(0,0,0)]\Phi_n(\mathbf{u})d\mathbf{u}. \tag{3.16}$$

For any $\varepsilon > 0$ can be chosen a $\delta > 0$ to satisfy

$$|\Psi(\mathbf{u}) - \Psi(0,0,0)| < \varepsilon/C_1, \tag{3.17}$$

where C_1 is the constant from Lemma 3.6 b). We represent $\Psi = \Psi_1 + \Psi_2$, such that

$$\|\Psi_1\|_2 \le \varepsilon/\sqrt{M_\delta} \quad \text{and} \quad \|\Psi_2\|_\infty < \infty, \tag{3.18}$$

where M_δ is the constant from Lemma 3.6 d). Using Lemma 3.6 b) - d) and (3.16) - (3.18) for sufficiently large n we obtain

$$\begin{aligned} |R_n| \ &\le \ \int_{\mathbb{E}_\delta} |\Psi(\mathbf{u}) - \Psi(\mathbf{0})||\Phi_n(\mathbf{u})|d\mathbf{u} + \int_{\mathbb{E}_\delta^c} |\Psi_1(\mathbf{u})||\Phi_n(\mathbf{u})|d\mathbf{u} \\ &+ \ \int_{\mathbb{E}_\delta^c} |\Psi_2(\mathbf{u}) - \Psi(\mathbf{0})||\Phi_n(\mathbf{u})|d\mathbf{u} \le \frac{\varepsilon}{C_1} \int_{\mathbb{E}_\delta} |\Phi_n(\mathbf{u})|d\mathbf{u} \\ &+ \ \|\Psi_1\|_2 \left[\int_{\mathbb{E}_\delta^c} \Phi_n^2(\mathbf{u})d\mathbf{u}\right]^{1/2} + C_2 \int_{\mathbb{E}_\delta^c} |\Phi_n(\mathbf{u})|d\mathbf{u} \le 3\,\varepsilon. \end{aligned}$$

This together with (3.16) implies (3.15). Lemma 3.7 is proved. □

4 Proofs

Proof of Theorem 2.1. For $f, g \in L^1(\mathbb{T})$ and $\mathbf{x} = (x_1, x_2, x_3, x_4)$ we set

$$F(\mathbf{x}) = f(x_1)f(x_2)g(x_3)g(x_4),$$

and let

$$H_n(\mathbf{x}) = G_n(x_1 - x_3)G_n(x_2 - x_3)G_n(x_4 - x_1)G_n(x_4 - x_2),$$

where

$$G_n(u) = \sum_{k=1}^{n} e^{iku} = e^{iu(n+1)/2} \cdot D_n(u). \tag{4.1}$$

It is easy to check that

$$\operatorname{tr}\left(T_n(f)T_n(g)\right)^2 = \int_{\mathbb{T}^4} F(\mathbf{x})H_n(\mathbf{x})d\mathbf{x}. \tag{4.2}$$

By Theorem B it is enough to consider the case $\alpha + \beta = \frac{1}{2}$. Thus, by Theorem C we need to prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}^4} F(\mathbf{x}) H_n(\mathbf{x}) d\mathbf{x} = 8\pi^3 \int_{\mathbb{T}} f^2(x) g^2(x) dx, \tag{4.3}$$

provided that

$$f(x) \le |x|^{-\alpha} L(x), \ |g(x)| \le |x|^{-\beta} L(x), \quad x \in \mathbb{T}, \tag{4.4}$$

where $L = L_1 + L_2 \in SV$ and

$$\alpha < 1, \ \beta < 1, \ \alpha + \beta = \frac{1}{2}, \quad \int_{\mathbb{T}} x^{-1} L^2(x) dx < \infty. \tag{4.5}$$

If $\alpha, \beta \ge 0$, then (4.4) implies $f \in L^{1/\alpha}(\mathbb{T})$, $g \in L^{1/\beta}(\mathbb{T})$, and Theorem 2.1 follows from Theorem A. Assuming $\beta < 0$, from (4.5) we have

$$\frac{1}{2} < \alpha < 1, \quad -\frac{1}{2} < \beta < 0. \tag{4.6}$$

For $\varepsilon \in (0,1)$ we set

$$f_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| < \varepsilon, \\ f(x), & \text{if } \varepsilon \le |x| \le \pi. \end{cases}$$

and

$$\mathbb{T}_{i,\varepsilon} = \left\{ \mathbf{x} \in \mathbb{T}^4 : |x_i| < \varepsilon \right\}, \quad i = 1, 2.$$

We have

$$\frac{1}{n} \int_{\mathbb{T}^4} F(\mathbf{x}) H_n(\mathbf{x}) d\mathbf{x} = J_n^1 + J_n^2,$$

where

$$J_n^1 := \frac{1}{n} \int_{\mathbb{T}^4} f_\varepsilon(x_1) f_\varepsilon(x_2) g(x_3) g(x_4) H_n(\mathbf{x}) d\mathbf{x}$$

and

$$|J_n^2| \le \frac{1}{n} \int_{\mathbb{T}_{1,\varepsilon}} |F(\mathbf{x}) H_n(\mathbf{x})| d\mathbf{x} + \frac{1}{n} \int_{\mathbb{T}_{2,\varepsilon}} |F(\mathbf{x}) H_n(\mathbf{x})| d\mathbf{x} =: I_n^1 + I_n^2.$$

Since $f_\varepsilon, g \in L^\infty(\mathbb{T})$ we have

$$\lim_{n\to\infty} J_n^1 = 8\pi^3 \int_{\mathbb{T}} f_\varepsilon^2(x) g^2(x) dx.$$

The last integral tends to $\int_{\mathbb{T}} f^2(x) g^2(x) dx$ as $\varepsilon \to 0$, hence (4.3) follows from

$$\lim_{\varepsilon\to 0, \ n\to\infty} \left(I_n^1 + I_n^2 \right) = 0. \tag{4.7}$$

It is enough to prove (4.7) for I_n^1. Set

$$B_{i,j} = \left\{ \mathbf{x} \in \mathbb{T}^4 : |x_i| \le \frac{|x_j|}{2} \right\}, \quad i = 1, 2, \ j = 3, 4,$$
$$B = \left\{ \mathbf{x} \in \mathbb{T}^4 : |x_1| < \varepsilon, \ |x_i| > \frac{|x_j|}{2}, \ i = 1, 2, \ j = 3, 4 \right\}.$$

Then we have

$$I_n^1 \le \frac{1}{n}\sum_{i=1}^{2}\sum_{j=3}^{4}\int_{B_{i,j}} F(\mathbf{x})H_n(\mathbf{x})d\mathbf{x} + \frac{1}{n}\int_B F(\mathbf{x})H_n(\mathbf{x})d\mathbf{x}. \tag{4.8}$$

Let $w \in SV$ be a function satisfying

$$\int_{\mathbb{T}} x^{-1}L^2(x)w^{-4}(x)dx < \infty. \tag{4.9}$$

Since $|x_3|/2 < |x_1 - x_3| < 2|x_3|$ if $\mathbf{x} \in B_{1,3}$, the bounds (4.4) and Lemma 3.4 imply

$$\begin{aligned} A_{1,3} : &= \frac{1}{n}\int_{B_{1,3}} F(\mathbf{x})G_n(\mathbf{x})d\mathbf{x} \\ &\le Cw^4\left(\frac{1}{n}\right)\int_{B_{1,3}} |x_1|^{-\alpha}|x_2|^{-\alpha}|x_3|^{-\beta}|x_4|^{-\beta}L(x_1)L(x_2)L(x_3)L(x_4) \\ &\times\ x_1 - x_3|^{-3/4}|x_2 - x_3|^{-3/4}|x_1 - x_4|^{-3/4}|x_2 - x_4|^{-3/4} \\ &\times\ w^{-1}(x_1 - x_3)w^{-1}(x_2 - x_3)w^{-1}(x_1 - x_4)w^{-1}(x_2 - x_4)d\mathbf{x} \end{aligned}$$

$$\begin{aligned} \le\ C\ & w^4\left(\frac{1}{n}\right)\int_{\mathbb{T}^2} |x_2|^{-\alpha}|x_4|^{-\beta}|x_2 - x_4|^{-3/4}L(x_2)L(x_4)w^{-1}(x_2 - x_4)dx_2 \\ \times & \int_{\mathbb{T}} |x_1|^{-\alpha}|x_1 - x_4|^{-3/4}L(x_1)w^{-1}(x_1 - x_4)dx_1 \\ \times & \int_{\mathbb{T}} T|x_3|^{-\beta-3/4}|x_2 - x_3|^{-3/4}L(x_3)w^{-1}(x_3)w^{-1}(x_2 - x_3)dx_3dx_2dx_4. \end{aligned}$$

Applying first Lemma 3.2, then Lemma 3.3 we obtain

$$\begin{aligned} A_{1,3} &\le Cw^4\left(\frac{1}{n}\right)\int_{\mathbb{T}^2} |x_2|^{-\alpha}|x_4|^{-\beta}|x_2 - x_4|^{-3/4}L(x_2)L(x_4)w^{-1}(x_2 - x_4) \\ &\times\ |x_4|^{-\alpha+1/4}L(x_4)w^{-1}(x_4)|x_2|^{-\beta-1/2}L(x_2)w^{-2}(x_2)dx_2dx_4 \\ &= Cw^4\left(\frac{1}{n}\right)\int_{\mathbb{T}} |x_4|^{-1/4}L^2(x_4)w^{-1}(x_4) \\ &\times \int_{\mathbb{T}} |x_2|^{-1}|x_2 - x_4|^{-3/4}L^2(x_2)w^{-2}(x_2)w^{-1}(x_2 - x_4)dx_2dx_4 \\ &\le Cw^4\left(\frac{1}{n}\right)\int_{\mathbb{T}} |x_4|^{-1}L^2(x_4)w^{-4}(x_4)dx_4 = o(1), \end{aligned} \tag{4.10}$$

as $n \to \infty$. Similarly we can prove that all the integrals in the first sum in (4.8) tend to zero as $n \to \infty$. To estimate the last integral in (4.8) we use (4.4) and Lemma 3.5

to obtain

$$\begin{aligned}
A: &= \frac{1}{n}\int_B |F(\mathbf{x})H_n(\mathbf{x})|d\mathbf{x}\\
&\le Cn^3 \int_B |x_1|^{-\alpha}|x_2|^{-\alpha}|x_3|^{-\beta}|x_4|^{-\beta}L(x_1)L(x_2)L(x_3)L(x_4)\\
&\times \psi_n(x_1-x_3)\psi_n(x_2-x_3)\psi_n(x_1-x_4)\psi_n(x_2-x_4)d\mathbf{x}\\
&\le Cn^3 \int\limits_{(-2\varepsilon,2\varepsilon)^2} |x_3|^{-1/2}|x_4|^{-1/2}L(x_3)L(x_4)\\
&\times \int_{\mathbb{T}} \psi_n(x_1-x_3)\psi_n(x_1-x_4)L(x_1)dx_1\\
&\times \int_{\mathbb{T}} \psi_n(x_2-x_3)\psi_n(x_2-x_4)L(x_2)dx_2dx_3dx_4
\end{aligned}$$

$$\le Cn \int\limits_{(-2\varepsilon,2\varepsilon)} |x_3|^{-1/2}L(x_3)\int_{\mathbb{T}}|x_4|^{-1/2}\psi_n^{1,5}(x_3-x_4)L(x_4)dx_4dx_3$$

$$\le C \int\limits_{-2n\varepsilon}^{2n\varepsilon} |y|^{-1/2}L\left(\frac{y}{n}\right)\int_{-\infty}^{\infty}\frac{|x|^{-1/2}}{(1+|x-y|)^{1,5}}L\left(\frac{x}{n}\right)dxdy. \tag{4.11}$$

Let us prove that

$$\int_{-\infty}^{\infty}\frac{|x|^{-1/2}}{(1+|x-y|)^{1,5}}L\left(\frac{x}{n}\right)dx \le Cy^{-1/2}L\left(\frac{y}{n}\right),\quad y\in\mathbb{T}. \tag{4.12}$$

Indeed, for $y\in\mathbb{T}$

$$\begin{aligned}
\int\limits_{|x|\le|y|}\frac{|x|^{-1/2}}{(1+|x-y|)^{1,5}}L\left(\frac{x}{n}\right)dx &\le CL\left(\frac{y}{n}\right)\int_{\mathbb{T}}|x|^{-1/2}dx\\
&\le CL\left(\frac{y}{n}\right)\le Cy^{-1/2}L\left(\frac{y}{n}\right).
\end{aligned} \tag{4.13}$$

According to Lemma 3.1 the function $t^{-1/2}L(t)$ is decreasing on some interval $(0,\delta)$. Hence, assuming without loss of generality, that $n>\frac{\pi}{\delta}$, we have for $|x|>|y|$

$$\begin{aligned}
|x|^{-1/2}L\left(\frac{x}{n}\right) = n^{-1/2}\left(\frac{|x|}{n}\right)^{-1/2}L\left(\frac{x}{n}\right) &\le n^{-1/2}\left(\frac{|y|}{n}\right)^{-1/2}L\left(\frac{y}{n}\right)\\
&= |y|^{-1/2}L\left(\frac{y}{n}\right).
\end{aligned}$$

Therefore

$$\int\limits_{|x|>|y|}\frac{|x|^{-1/2}}{(1+|x-y|)^{1,5}}L\left(\frac{x}{n}\right)dx \le C|y|^{-1/2}L\left(\frac{y}{n}\right)\int_{-\infty}^{\infty}\frac{1}{(1+|x|)^{1,5}}dx$$

$$\leq C|y|^{-1/2} L\left(\frac{y}{n}\right). \tag{4.14}$$

From (4.13), (4.14) we obtain (4.12) and from (4.11), (4.12) and (4.5)

$$A \leq C \int\limits_{-2n\varepsilon}^{2n\varepsilon} |y|^{-1} L^2\left(\frac{y}{n}\right) dy = C \int\limits_{-2\varepsilon}^{2\varepsilon} |t|^{-1} L^2(t) dt = o(\varepsilon), \tag{4.15}$$

as $\varepsilon \to 0$. A combination of (4.8), (4.10) and (4.15) yields (4.7). Theorem 2.1 is proved. □

Proof of Theorem 2.2. By the change of variables $x_1 = u$, $x_1 - x_3 = u_1$, $x_3 - x_2 = u_2$ and $x_2 - x_4 = u_3$ from (4.2) we obtain

$$\begin{aligned} \operatorname{tr}(T_n(f)T_n(g))^2 \;&=\; \int_{T^4} G_n(u_1)G_n(u_2)G_n(u_3)G_n(-u_1-u_2-u_3) \\ &\times\; f(u)g(u-u_1)f(u-u_1-u_2)g(u-u_1-u_2-u_3)\, du_1du_2du_3du_4, \end{aligned} \tag{4.16}$$

where $G_n(u)$ is as in (4.1). Taking into account the equality

$$e^{iu_1(n+1)/2} \cdot e^{iu_2(n+1)/2} \cdot e^{iu_3(n+1)/2} \cdot e^{-i(u_1+u_2+u_3)(n+1)/2} = 1$$

and that $D_n(u)$ is even function, from (4.16) we obtain

$$\operatorname{tr}(T_n(f)T_n(g))^2 = 8\pi^3 \int_{\mathbb{T}^3} \Psi(u_1,u_2,u_3)\Phi_n(u_1,u_2,u_3)\, du_1du_2du_3, \tag{4.17}$$

where $\Phi_n(u_1,u_2,u_3)$ is defined by (3.7), $\Psi(u_1,u_2,u_3) = \varphi(u_1,u_1+u_2,u_1+u_2+u_3)$ and $\varphi(u_1,u_2,u_3)$ is defined by (2.4). By Theorem C and (4.17) we need to prove that

$$\lim_{n\to\infty} \int_{\mathbb{T}^3} \Psi(\mathbf{u})\Phi_n(\mathbf{u})d\mathbf{u} = \int_{\mathbb{T}} f^2(x)g^2(x)dx. \tag{4.18}$$

Now, since the functions $\varphi(u_1,u_2,u_3)$ and $\Psi(u_1,u_2,u_3) = \varphi(u_1,u_1+u_2,u_1+u_2+u_3)$ are square integrable and continuous at $(0,0,0)$ simultaneously, and

$$\Psi(0,0,0) = \int_{\mathbb{T}} f^2(x)g^2(x)dx,$$

from Lemma 3.7 we obtain (4.18). Theorem 2.2 is proved. □

Proof of Proposition 2.1. To show that Theorem 2.2 implies Theorem A it is enough to prove that the function

$$\varphi(\mathbf{t}) := \int_{\mathbb{T}} f_0(u)f_1(u-t_1)f_2(u-t_2)f_3(u-t_3)du, \quad \mathbf{t} = (t_1,t_2,t_3) \tag{4.19}$$

belongs to $L^2(\mathbb{T}^3)$ and is continuous at $(0,0,0)$, provided that

$$f_i \in L^{p_i}(\mathbb{T}), \quad 1 \leq p_i \leq \infty, \quad i = 0,1,2,3, \qquad \sum_{i=0}^{3} \frac{1}{p_i} \leq 1. \tag{4.20}$$

It follows from Hölder inequality and (4.20) that

$$|\varphi(\mathbf{t})| \leq \prod_{i=0}^{3} ||f_i||_{L^{p_i}(\mathbb{T})}, \quad \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{T}^3.$$

Therefore, $\varphi(\mathbf{t}) \in L^2(\mathbb{T}^3)$. To prove the continuity of $\varphi(\mathbf{t})$ at the point $(0,0,0)$ we consider three cases.

Case 1. $p_i < \infty, \quad i = 0, 1, 2, 3.$
For an arbitrary $\varepsilon > 0$ we find $\delta > 0$ satisfying (see (4.20))

$$\|f_i(u-t) - f_i(u)\|_{p_i} \leq \varepsilon, \quad i = 1, 2, 3, \quad \text{i}f \quad |t| \leq \delta. \tag{4.21}$$

We fix $\mathbf{t} = (t_1, t_2, t_3)$ with $|\mathbf{t}| < \delta$ and denote

$$\overline{f}_i(u) = f_i(u + t_i) - f_i(u), \quad i = 1, 2, 3.$$

Then (4.21) implies $\|\overline{f}_i\|_{p_i} \leq \varepsilon$, $i = 1, 2, 3$ and we have

$$\varphi(\mathbf{t}) = \int_{\mathbb{T}} f_0(u) \prod_{i=1}^{3} \left(\overline{f}_i(u) + f_i(u)\right) du = \varphi(0,0,0) + W,$$

where the quantity W is a sum of five integrals. Each of them contains at least one function $\overline{f}_i$ and can be estimated as the following one

$$\left| \int_{\mathbb{T}} f_o(u) \overline{f}_1(u) f_2(u) f_3(u) du \right| \leq \|f_o\|_{p_0} \|\overline{f}_1\|_{p_1} \|f_2(u)\|_{p_2} \|f_3\|_{p_3}\| \leq A\varepsilon.$$

Case 2. $p_i \leq \infty, \ i = 0, 1, 2, 3, \ \sum\limits_{i=0}^{3} \frac{1}{p_i} < 1.$
There exist finite numbers $p_i' < p_i, \ i = 0, 1, 2, 3, \ \sum_{i=0}^{3} 1/p_i' \leq 1$ for which we have $f_i \in L^{p_i}$. Hence φ is continuous at $(0,0,0)$ as in the case 1.

Case 3. $p_i \leq \infty, \ i = 0, 1, 2, 3, \ \sum\limits_{i=0}^{3} \frac{1}{p_i} = 1.$
At least one of numbers p_i is finite. Suppose, without loss of generality, that $p_0 < \infty$. For any $\varepsilon > 0$ we find functions f_0', f_0'' such that

$$f_0 = f_0' + f_0'', \quad f_0' \in L^\infty, \quad \|f_0''\|_{p_0} < \varepsilon. \tag{4.22}$$

Then

$$\varphi(\mathbf{t}) = \varphi'(\mathbf{t}) + \varphi''(\mathbf{t}), \tag{4.23}$$

where the functions φ' and φ'' are defined as φ in (4.19) with f_0 replaced by f_0' and f_0'' respectively. From (4.22) follows that φ' is continuous at $(0,0,0)$ (see case 2), while for φ'' the Hölder inequality imply $|\varphi''(\mathbf{t})| \leq A\varepsilon$. Hence, for sufficiently small $|\mathbf{t}|$

$$|\varphi(\mathbf{t}) - \varphi(0,0,0)| \leq |\varphi'(\mathbf{t}) - \varphi'(0,0,0)| + |\varphi''(\mathbf{t}) - \varphi''(0,0,0)| \leq (A+1)\varepsilon,$$

and the result follows.

Now proceed to prove that Theorem 2.2 implies Theorem D. To this end it is enough to show that the function

$$\varphi(\mathbf{t}) = \int_{\mathbb{T}} f(u)g(u-t_1)f(u-t_2)g(u-t_3)du, \quad \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{T}^3$$

is continuous at $(0,0,0)$, provided that f and g satisfy conditions of theorem D, i. e. $f \in L_2(\mathbb{T})$, $g \in L_2(\mathbb{T})$, $fg \in L_2(\mathbb{T})$ and (1.11) holds.
Since

$$\varphi^2(\mathbf{t}) \le 2\pi \int_{\mathbb{T}} f^2(u)g^2(u-t_1)f^2(u-t_2)g^2(u-t_3)du,$$

we have

$$\int_{\mathbb{T}^3} \varphi^2(\mathbf{t})\, d\mathbf{t} \le \int_{\mathbb{T}} \left[\int_{\mathbb{T}} g^2(u-t_1)\, dt_1 \int_{\mathbb{T}} f^2(u-t_2)\, dt_2 \int_{\mathbb{T}} g^2(u-t_3)\, dt_3 \right]$$
$$\times f^2(u)du = ||f||_2^4 ||g||_2^4 < \infty.$$

Now we prove the continuity of $\varphi(\mathbf{t})$ at the point $(0,0,0)$. Let ε be an arbitrary positive number. We denote

$$E_K = \{u \in \mathbb{T} : |f(u)| \le K\}, \quad f_1(u) = \chi_{E_K}(u) f(u), \quad f_2(u) = f(u) - f_1(u),$$

where $K > 0$ is chosen to satisfy $\int_{\mathbb{T}\setminus E_k} f^2(u)g^2(u)du \le \varepsilon$. Then

$$f = f_1 + f_2, \quad \|f_1\|_\infty \le K, \quad \int_{\mathbb{T}} f_2^2(u)g^2(u)du \le \varepsilon. \tag{4.24}$$

We have

$$\begin{aligned} \varphi(\mathbf{t}) &= \int_{\mathbb{T}} f_1(u)g(u-t_1)f_1(u-t_2)g(u-t_3)du \\ &+ \int_{\mathbb{T}} f_2(u)g(u-t_1)f(u-t_2)g(u-t_3)du \\ &+ \int_{\mathbb{T}} f_1(u)g(u-t_1)f_2(u-t_2)g(u-t_3)du \\ &=: \varphi_1(\mathbf{t}) + \varphi_2(\mathbf{t}) + \varphi_3(\mathbf{t}). \end{aligned} \tag{4.25}$$

We estimate the functions $\varphi_k(\mathbf{t})$, $k = 1, 2, 3$ separately. We have

$$\begin{aligned} \varphi_1(\mathbf{t}) &= \int_{\mathbb{T}} f_1(u)g(u-t_1)f_1(u-t_2)\left[g(u-t_3) - g(u)\right] du \\ &+ \int_{\mathbb{T}} f_1(u)g(u)f_1(u-t_2)\left[g(u-t_1) - g(u)\right] du \\ &+ \int_{\mathbb{T}} f_1(u)g^2(u)f_1(u-t_2)du =: I_1 + I_2 + I_3. \end{aligned} \tag{4.26}$$

Using Hölder inequality, from (4.24) we get

$$|I_1| \le K^2 \|g\|_2 \cdot \|g(u+t_3) - g(u)\|_2 = o(1), \quad \text{as} \quad t_3 \to 0. \tag{4.27}$$

Similarly

$$|I_2| = o(1) \quad \text{as} \quad t_1 \to 0. \tag{4.28}$$

From (4.24) we have

$$\begin{aligned}
\left| I_3 - \int_{\mathbb{T}} \varphi(0,0,0) \right| &= \left| \int_{\mathbb{T}} f_1(u+t_2) g^2(u+t_2) f_1(u) du - \int_{\mathbb{T}} f_1^2(u) g^2(u) du \right| \\
&+ \left| \int_{\mathbb{T}} f_2^2(u) g^2(u) du \right|
\end{aligned}$$

$$\leq K \left\| f_1(u+t_2) g^2(u+t_2) - f_1(u) g_1^2(u) \right\|_1 + \varepsilon = o(1) + \varepsilon, \tag{4.29}$$

as $t_2 \to 0$. From (4.26)-(4.29) for sufficiently small $|\mathbf{t}|$ we obtain

$$|\varphi_1(\mathbf{t}) - \varphi(0,0,0)| \leq 2\varepsilon. \tag{4.30}$$

Next, for $\varphi_2(\mathbf{t})$ we have

$$\begin{aligned}
|\varphi_2(\mathbf{t})|^2 &\leq \int_{\mathbb{T}} f_2^2(u) g^2(u-t_1) du \int_{\mathbb{T}} f_2^2(u-t_2) g^2(u-t_3) du \\
&= \left| \int_T f^2(u) g^2(u-t_1) du - \int_T f_1^2(u) g^2(u-t_1) du \right| \\
&\times \int_{\mathbb{T}} f^2(u) g^2(u+t_2-t_3) du \\
&\to \left| \int_{\mathbb{T}} f^2(u) g^2(u) du - \int_{\mathbb{T}} f_1^2(u) g^2(u) du \right| \int_{\mathbb{T}} f^2(u) g^2(u) du.
\end{aligned}$$

as $|\mathbf{t}| \to 0$. Therefore, in view of (4.24) for sufficiently small $|\mathbf{t}|$

$$|\varphi_2(\mathbf{t})| \leq \varepsilon \int_{\mathbb{T}} f^2(u) g^2(u) du. \tag{4.31}$$

Similarly we can prove that for enough small $|\mathbf{t}|$

$$|\varphi_3(\mathbf{t})| \leq \varepsilon \int_{\mathbb{T}} f^2(u) g^2(u) du. \tag{4.32}$$

A combination of (4.25) and (4.30)-(4.32) yields

$$\lim_{\mathbf{t} \to 0} \varphi(\mathbf{t}) = \varphi(0,0,0).$$

This completes the proof of Proposition 2.1. □

Proof of Proposition 2.2. We construct functions $f(\lambda)$ and $g(\lambda)$ satisfying the conditions (2.5) and (2.6). Let $p \geq 2$ be fixed, we choose a number $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} > 1$. For such p and q consider the functions $f_0(\lambda)$ and $g_0(\lambda)$ defined by (1.12) and (1.13) respectively. For an arbitrary finite positive constant C we set $g_\pm(\lambda) = g_0(\lambda) \pm C$. Since the functions $f_0(\lambda)$ and $g_0(\lambda)$ have disjoint supports, we have

$$\int_{-\pi}^{\pi} f_0^2(\lambda)\, g_\pm^2(\lambda)\, d\lambda = \int_{-\pi}^{\pi} f_0^2(\lambda)\, (g_0 \pm C)^2(\lambda)\, d\lambda = C^2 \int_{-\pi}^{\pi} f_0^2(\lambda)\, d\lambda < \infty,$$

and hence (2.5) is fulfilled. Next, by (1.14)

$$\frac{1}{n}\ \mathrm{tr}\,(T_n(f_0)T_n(g_0))^2 \longrightarrow \infty \quad \text{as} \quad n\to\infty, \tag{4.33}$$

and by Theorem A with $p_1 = p \geq 2$ and $p_2 = \infty$,

$$\frac{1}{n}\ C^2\ \mathrm{tr}\left(T_n^2(f_0)\right) \longrightarrow 8\pi^3 C^2 \int_{-\pi}^{\pi} f_0^2(\lambda)\,d\lambda < \infty. \tag{4.34}$$

On the other hand, we have

$$\mathrm{tr}\,(T_n(f_0)T_n(g_\pm))^2 = \ \mathrm{tr}\,(T_n(f_0)T_n(g_0\pm C))^2$$

$$= \ \mathrm{tr}\,(T_n(f_0)T_n(g_0))^2 \pm 2C\ \ \mathrm{tr}\left(T_n^2(f_0)T_n(g_0)\right) + C^2\ \mathrm{tr}\left(T_n^2(f_0)\right),$$

which combined with (4.33) and (4.34) implies

$$\frac{1}{n}\ \mathrm{tr}\,(T_n(f_0)T_n(g_+))^2 + \frac{1}{n}\ \mathrm{tr}\,(T_n(f_0)T_n(g_-))^2$$

$$= \frac{2}{n}\ \mathrm{tr}\,(T_n(f_0)T_n(g_0))^2 + \frac{2}{n}C^2\ \mathrm{tr}\left(T_n^2(f_0)\right) \to \infty \quad \text{as} \quad n\to\infty.$$

Therefore, either

$$\lim_{n\to\infty}\sup \frac{1}{n}\ \mathrm{tr}\,(T_n(f_0)T_n(g_+))^2 = \infty,$$

or

$$\lim_{n\to\infty}\sup \frac{1}{n}\ \mathrm{tr}\,(T_n(f_0)T_n(g_-))^2 = \infty.$$

Thus, we obtain

$$\lim_{n\to\infty}\sup \chi_2(\widetilde{Q}_n) = \lim_{n\to\infty}\sup \frac{2}{n}\ \mathrm{tr}\,(T_n(f)T_n(g))^2 = \infty$$

with $f = f_0$ and $g = g_+$ or $g = g_-$. This completes the proof of Proposition 2.2. □

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