Institute of Mathematics

NATIONAL ACADEMY OF SCIENCES OF ARMENIA

Weighted Classes of Regular Functions Area Integrable Over the Unit Disc

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Preprint 2002 – 01

December 12, 2002

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Abstract. This preprint contains some generalizations of the main theorems of M.M.Djrbashian of 1945–1948 which laid ground for the theory of A^p_α (or initially $H^p(\alpha)$) spaces and his factorization theory of classes $N\{\omega\}$ exhausting all functions meromorphic in the unit disc. Also some later results on A^p_α spaces and Nevanlinna's weighted class are improved.

The preprint contains the main analytic apparatus for generalizing almost all known results on A^p_α spaces within a new theory, where instead $(1-r^2)^\alpha dr$ $(-1<\alpha<+\infty$, $0<r<1)$ some weights of the form $d\omega(r^2)$ are used. The obtained results make evident that the theory of A^p_ω spaces and the factorization theory of M.M.Djrbashian are inseparable parts of a general theory of classes of regular functions associated with M.M.Djrbashian general integrodifferentiation. The author hopes that the publication of this preprint can lead to clarification of some priority misunderstandings in the field.

Introduction.

The earliest work of M.M.Djrbashian [1] published in 1945 (see also [2] containing the proofs of the representations of [1] and some additional theorems) was mainly aimed to improve Nevanlinna's result (see [3], Sec. 216) on the density of zeros and poles of functions $f(z)$ meromorphic in the unit disc, for which the Riemann–Liouville fractional integral of the growth characteristic $T(r,f)$ is bounded, i.e.

$$\int_0^1 (1-r)^\alpha T(r,f)dr < +\infty \tag{1}$$

for a given $\alpha > -1$. The same work [1] contains investigation of the similar Hardy type classes $H^p(\alpha)$ of holomorphic functions in $|z|<1$, for which the Riemann–Liouville fractional integral of

$$M_p^p(r,f) = \int_0^{2\pi} |f(re^{i\vartheta})|^p d\vartheta$$

is bounded, i.e.

$$H^p(\alpha) \equiv A^p_\alpha : \qquad \iint_{|\zeta|<1} (1-|\zeta|)^\alpha |f(\zeta)|^p d\sigma(\zeta) < +\infty, \tag{2}$$

where $\alpha \in (-1,+\infty)$ and $p \in [1,+\infty)$ are any fixed numbers and $d\sigma(\zeta)$ is the Lebesgue's area measure. Particularly, in [1] the representation formula of these classes was first obtained. It has to be mentioned, that $H^p(\alpha) \equiv A^p_\alpha$ is a generalization of the non-weighted Hilbert space

$$H^2(0) \equiv A^2_0 : \qquad \iint_{|\zeta|<1} |f(\zeta)|^2 d\sigma(\zeta) < +\infty$$

which was widely known for a long time. The oldest known work, where the classes $H^2(0) \equiv A_0^2$ ($\equiv A^2$) were introduced and investigated was authored by L.Biberbach[4] (1914) to whom in all probability these classes are to be attributed (see [5], pp. 44, 150, 322)[1].

Later a new approach to application of fractional integrodifferentiation led M.M.Djrbashian (see [11], Ch. IX and [12, 13, 14]) to the factorization theory of his Nevanlinna type $N\{\omega\}$ classes *the sum of which coincides with the whole set of functions meromorphic in* $|z| < 1$. This theory is given in completed form in his joint monograph with V.S.Zakarian [15]. The classes $N\{\omega\}$ were introduced in [13] by the use of a generalization of the Riemann–Liouville operator [12] which is written in the form

$$L_\omega \log|f(z)| = -\int_0^1 \log|f(tz)|d\omega(t), \quad |z| < 1,$$

if the function $\omega(x)$ satisfies some natural conditions.

Further development of investigation methods led the author of this paper [16] to an extension of M.M.Djrbashian's factorization theory to the set of all functions δ-subharmonic in $|z| < 1$, where the theory got an extremely explicit interpretation. Namely, the replacement of $\log|f(z)|$ by an arbitrary function $u(z)$ which is δ-subharmonic in $|z| < 1$ led to the following understanding of M.M.Djrbashian T_ω characteristics and N_ω classes:

$$T_\omega(r,u) = T(r, L_\omega u) \quad \textit{and hence} \quad u \in N_\omega \quad \textit{if and only if} \quad L_\omega u(z) \in N,$$

i.e. Nevanlinna characteristic of the δ-subharmonic function $L_\omega u(z)$ is bounded.

Nevertheless, the old results of M.M.Djrbashian [1, 2] still remain in considerable interest as they find development and application in numerous contemporary investigations (see the monographs by A.E.Djrbashian–F.A.Shamoian[17] and H.Hedenmalm–B.Korenblum–K.Zhu [18] containing large lists of references, see also [19 – 25][2]

This paper is aimed in generalization of the main theorems of M.M.Djrbashian [1, 2] which in essence gave rise to the theory of A^p_α spaces in the unit disc and to his factorization theory of functions meromorphic in the disc [13, 14]. Besides, similar generalizations of some later results of other authors, related to the spaces A^p_α and to the weighted Nevanlinna class (1) are obtained. Instead $(1-r^2)^\alpha dr$ ($-1 < \alpha < +\infty$, $0 < r < 1$), some weights of the form $d\omega(r^2)$ are used[3]. This unites the results of the paper with the factorization theory of M.M.Djrbashian.

This paper improves the results of the report [20]. The author thanks K.L.Avetisyan for Lemma 1.1 and some additions to References. Besides, the author is thankful to S.G.Rafaelyan for a remark which laid ground for Section 7.

1. The spaces A^p_ω and their representations

1.1. We define A^p_ω as the set of all functions $f(z)$ holomorphic in $|z| < 1$, for which

$$\|f\|_{p,\omega} = \left\{ \frac{1}{2\pi} \iint_{|\zeta|<1} |f(\zeta)|^p |d\mu_\omega(\zeta)| \right\}^{1/p} < +\infty, \qquad 0 < p < \infty, \tag{1.1}$$

[1] for A_0^2, see also Carleman[6] (1922), S.Bergman[7] (1929), W.Wirtinger[8] (1932). Still without the representation formula of A^p_α, Hardy–Littlewood[9] (1932) investigated the fractional integration operator in A^p_α, and M.V.Keldysch [10] (1941) proved some approximation theorems in A^p_α.

[2] one can see that several assertions on the historical background of A^p_α spaces (see, for instance, pp. 6 – 8 in [17]) are neglected in some publications, which causes new misunderstandings.

[3] in 1980's F.A.Shamoian had proved some less general results in the field.

where $d\mu_\omega(\rho e^{i\vartheta}) = -d\omega(\rho^2)d\vartheta$ and $\omega(t) \in \Omega_A$, i.e. is defined in $[0,1]$ and such that:

$$\textbf{(i)} \qquad 0 < \bigvee_\delta^1 \omega < \infty \qquad \text{for any} \quad \delta \in [0,1),$$

$$\textbf{(ii)} \qquad \Delta_n \equiv \Delta_n(\omega) = -\int_0^1 t^n d\omega(t) \neq 0, \qquad n = 1,2,\dots,$$

$$\textbf{(iii)} \qquad \liminf_{n\to\infty} \sqrt[n]{|\Delta_n|} \geq 1\,.$$

The similar class L^p_ω is assumed to be defined in the same way, except the holomorphity requirement. Note that $\limsup_{n\to\infty} \sqrt[n]{|\Delta_n|} \leq \lim_{n\to\infty} \sqrt[n]{\bigvee_\delta^1 \omega} = 1$. Thus, $\lim_{n\to\infty} \sqrt[n]{|\Delta_n|} = 1$ under **(i)** and **(iii)**.

Proposition 1.1. *For any fixed $p \in (0,\infty)$ the sum $\bigcup_{\omega\in\Omega_A} A^p_\omega$ coincides with the set of all functions holomorphic in $|z| < 1$.*

Proposition 1.2. *For any $p \in [1,\infty)$ and $\omega \in \Omega_A$ the class A^p_ω is a Banach space with the norm (1.1), and A^p_ω becomes A^p_α when $\omega(x) = (1-x)^{1+\alpha}$ $(-1 < \alpha < \infty)$.*

1.2. We start by an estimate which is of independent interest.

Lemma 1.1. *If $f(z) \in A^p_\omega$, where $\omega(x) \in \Omega_A$ and $0 < p < +\infty$ are arbitrary. Then*

$$|f(z)|^p \leq \frac{2\|f\|^p_{p,\omega}}{(\rho - |z|)\int_{\rho^2}^1 |d\omega(x)|}, \qquad |z| < \rho < 1.$$

The following assertion is a generalization of Theorem I in [2].

Theorem 1.1. *If $f(z) \in A^p_\omega$ for some $\omega \in \Omega_A$ and $p \in [1,\infty)$, then the following formulas are true in $|z| < 1$:*

$$f(z) = \frac{1}{2\pi}\iint_{|\zeta|<1} f(\zeta) C_\omega(z\bar\zeta) d\mu_\omega(\zeta), \tag{1.2}$$

$$f(z) = -\overline{f(0)} + \frac{1}{\pi}\iint_{|\zeta|<1} \left\{\operatorname{Re} f(\zeta)\right\} C_\omega(z\bar\zeta) d\mu_\omega(\zeta), \tag{1.2'}$$

where $C_\omega(z) = \sum_{k=0}^\infty z^k/\Delta_k$ $(\Delta_0 = 1)$ is the Cauchy type kernel of M.M. Djrbashian [12, 13, 14, 15].

1.3. Under some additional requirements on $\omega \in \Omega_A$, the following descriptive representation is true for several A^2_ω, the sum of which still is equal to the set of all functions holomorphic in $|z| < 1$.

Theorem 1.2. *Let $\omega \in \Omega_A$ be continuously differentiable in* $[0.1]$ *and such that $\omega \searrow$, $\omega(1) \equiv \omega(1-0) = 0$ and $\omega(0) = 1$. Further, let $\widetilde\omega(x)$ be the Volterra square of $\omega(x)$, i.e.*

$$\widetilde\omega(x) = -\int_x^1 \omega\Big(\frac{x}{\sigma}\Big) d\omega(\sigma), \qquad 0 < x < 1.$$

Then $\widetilde\omega \in \Omega_A$ and $A^2_{\widetilde\omega}$ coincides with the set of all functions which are representable in the form

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\vartheta}) C_\omega(ze^{-i\vartheta}) d\vartheta, \qquad |z| < 1, \quad \varphi(e^{i\vartheta}) \in L^2[0,2\pi]. \tag{1.3}$$

For any $f(z) \in A^2_{\tilde\omega}$ there exists a unique function $\varphi_0(z)$ of the ordinary Hardy space H^2, such that (1.3) is true with $\varphi_0(e^{i\vartheta})$. This function is found by the formula

$$\varphi_0(z) = L_\omega f(z) = -\int_0^1 f(tz)d\omega(t), \qquad |z| < 1. \tag{1.4}$$

Besides, $\|\varphi_0\|_{H^2} = \|f\|_{2,\tilde\omega}$ and $\varphi - \varphi_0 \perp H^2$ for any $\varphi(e^{i\vartheta}) \in L^2[0,2\pi]$ for which (1.3) is true. The operator L_ω is an isometry $A^2_{\tilde\omega} \longrightarrow H^2$, and the integral (1.3) defines $(L_\omega)^{-1}$ on H^2.

Remark 1.1. For $\omega(x) = (1-x)^{(1+\alpha)/2}$ $(\alpha > -1)$ the previous Theorem 1.2 becomes, in essence, the union of Theorems IV and V of [2]. For the Biberbach space A^2_0 $(\alpha = 0)$ the representation (1.3) with φ_0 of the form (1.4) first has been proved by M.V. Keldysch (see [2]).

1.4. A representation over the boundary of the disc is true also in the general case $1 \le p < +\infty$. It is natural to call this Shamoian representation as such representations were first found in [21].

Theorem 1.3. *Let $1 \le p < +\infty$ and let $\omega(x) \in \Omega_A$ and $\bigvee_0^1 \omega = 1$. Then:*

1°. *Any function $f(z) \in A^p_\omega$ is representable in the form*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} C_{\omega_1}(ze^{-i\vartheta})\varphi(e^{i\vartheta})d\vartheta, \quad |z| < 1, \tag{1.5}$$

where $\omega_1(x) = \omega(x^2)$ and $\varphi(z) = L_{\omega_1} f(z) \in H^p$.

2°. *In A^p_ω we have $\|L_{\omega_1}\| \le 1$, and (1.5) represents $L^{-1}_{\omega_1}$ in $L_{\omega_1} A^p_\omega$.*

Remark 1.2. One can verify that Theorem 1.3 remains true if $\omega_1(x)$ is replaced by $\omega(x)$ itself.

Remark 1.3. For the particular case $\omega(x) = (1-x)^{1+\alpha}$ $(-1 < \alpha < +\infty)$ some descriptive representations of A^p_ω by integrals over the unit circle were established by F.A.Shamoian [21] for $p = 1$. A development of the methods of [21] in [26] led to similar representations of A^p_α for $p \ge 1$. These representations establish one-to-one correspondences between A^p_α and definite O.V.Besov spaces. Similar to the results of [21] and [26], the assertion of Theorem 1.3 evidently is true with

$$\omega_\varepsilon(x) = -\int_x^1 \varepsilon\left(\frac{x}{t}\right) d\omega(t), \quad 0 < x < 1,$$

and L_{ω_ε}, C_{ω_ε} in the representation (1.5) of $f(z) \in A^p_\omega$, provided $\varepsilon(x) \in \Omega_A$ is continuously differentiable in $[0,1)$ and such that $\varepsilon(x) \searrow$ and $\varepsilon(1-0) = 0$.

2. The spaces $A^p_\omega \subset H^p$

Below we shall consider some Dirichlet type weighted spaces that we again denote A^p_ω due to their similarity to those $N\{\omega\}$ classes of M.M.Djrbashian, which are contained in Nevanlinna's class N, namely these A^p_ω are contained in Hardy's H^p and have the same boundary property as $N\{\omega\} \subset N$.

Definition 2.1. *The class $\widetilde\Omega_A$ is the set of those $\omega(x)$ which are continuous and nondecreasing in $[0,1)$ and such that $\int_0^1 \omega(x)dx = 1$ and $|\omega(x) - 1| \le Kx$ $(0 \le x < \delta)$ for some $K > 0$ and $\delta \in (0,1)$.*

In contrast to the definition of Ω_A, this definition permits a function $\omega(x) \in \widetilde\Omega_A$ to tend to $+\infty$ as $x \to +0$ and to introduce:

Definition 2.2. A^p_ω $(0 < p < +\infty)$ *is the set of those functions* $f(z)$ *for which* $f'(z) \in A^p_{\widetilde{\omega}}$, *where* $\widetilde{\omega}(x) = \int_x^1 \omega(t)dt$ *and* $\omega(x) \in \widetilde{\Omega}_A$.

One can verify that the last definition is correct since it provides $\widetilde{\omega}(x) \in \Omega_A$. Besides, it is obvious that A^p_ω $(1 \le p < +\infty,\ \omega(x) \in \widetilde{\Omega}_A)$ is a Banach space with the norm $\|f\|_{p,\omega} = \|f'\|_{p,\widetilde{\omega}}$.

Before giving the theorem, we have to recall the concept of ω–capacity introduced by M.M. Djrbashian and V.S.Zakarian [27, 28, 14, 15] as a generalization of Frostman's well known α–capacity (holding for $\omega(x) = (1-x)^\alpha$, $-1 < \alpha < 0$). Assuming that $\omega(x) > 0$ is continuous in $[0,1)$, $|\omega(x) - 1| \le Kx$ $(0 \le x < \delta)$ for some $K > 0$ and $\delta \in (0,1)$, $\omega(0) = 1$ and $\int_0^1 \omega(x)dx < +\infty$, and

$$C(z;\omega) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k\int_0^1 t^{k-1}\omega(x)dx},$$

they introduced

Definition 2.3. *A Borel set* $E \subset [0, 2\pi]$ *has positive* ω*–capacity if there exists a nonnegative measure* $\mu \prec E$ *for which*

$$\sup_{|z|\le 1} \int_0^{2\pi} |C(ze^{-i\vartheta};\omega)|d\mu(\vartheta) < +\infty. \tag{2.1}$$

Otherwise (i.e. if the integral (2.1) is unbounded for any $\mu \prec E$*)* E *has zero* ω*–capacity.*

Theorem 2.1. *Let* $1 \le p < +\infty$ *and let* $\omega(x) \in \widetilde{\Omega}_A$. *Then:*

1°. $A^p_\omega \subset H^p$

2°. *For any* $f(z) \in A^p_\omega$ *the finite nontangential boundary values* $f(e^{i\vartheta})$ *exist for all* $\vartheta \in [0, 2\pi]$ *except, perhaps, for a set* $E \subset [0, 2\pi]$ *of zero* ω*–capacity.*

Remark 2.1. Using the representation (1.3) of Theorem 1.2, one can prove a more sharp theorem on the boundary properties of the classes $A^2_\omega \subset H^2$.

3. Evaluation of M.M.Djrbashian ω-kernels

This section contains the estimates of the still unpublished work [29], which were essential for proving further results.

As M.M.Djrbashian had often stated, one of the most significant problems related with his factorization theory [13, 14, 15] is the evaluation of C_ω (and hence of $S_\omega = 2C_\omega - 1$) kernels which depend on a function-parameter $\omega(x)$ given in $(0,1)$ and are used in the representations of this theory (see also [16], where the theory is extended to all functions δ–subharmonic in $|z| < 1$). Later, for solution of some representation problems in tube domains A.H.Karapetyan [30] constructed some kernels which in the one-dimensional case are the half-plane similarities of C_ω and take the form

$$C_\omega(z) \equiv \int_0^{+\infty} \frac{e^{izt}dt}{t\int_0^{+\infty} e^{-\sigma t}\omega(\sigma)d\sigma}, \qquad z \in G^+ = \{z : \operatorname{Im} z > 0\} \tag{3.1}$$

(we use the same notation for these new kernels and preassume some restrictions on $\omega(t)$, $0 < t < +\infty$, providing the holomorphity of $C_\omega(z)$ in G^+). The main assumption was that under some additional conditions on the behavior of the parameter-functions $\omega(x)$ in $(0,1)$ and $(0,+\infty)$ the following estimates have to be true:

$$|C_\omega(z)| \le \frac{M}{|z^2\omega'(y)|}, \quad z = x + iy \in G^+ \quad \text{and} \quad |C_\omega(z)| \le \frac{M}{|(1-z)^2\omega'(|z|)|}, \quad |z| < 1, \tag{3.2}$$

which are "almost" generalizations of the representations

$$C_\omega(z)\Big|_{\omega(x)=x^\alpha} = \frac{1}{(-iz)^{1+\alpha}}, \ z \in G^+ \quad \text{and} \quad C_\omega(z)\Big|_{\omega(x)=(1-x)^\alpha} = \frac{1}{(1-z)^{1+\alpha}}, \ |z|<1 \ (\alpha > -1).$$

3.1. We use a united evaluation method for proving (3.2) for both kernels, under some conditions in which the derivatives of $\omega(x)$ in $(0,1)$ (or $(0,+\infty)$) decrease as $x \to 1-0$ (or $x \to +0$) not more rapidly than the function $(1-x)^\alpha$ (or x^α). These estimates are exact on the positive radius $z = r \in (0,1)$ and on the imaginary half-axis $z = iy$, $y \in (0,+\infty)$. The "model" case is the evaluation of the half-plane kernel (3.1), and the main technical tool is the following, perhaps known, similarity of Abel's theorem.

Lemma 3.1. *Let $\varphi(t) > 0$ be a function defined in $(0,+\infty)$.*

1°. *If $\varphi(t) \nearrow$ but $t^{-\alpha}\varphi(t) \searrow$ in $(0,+\infty)$ for an $\alpha > 0$, then*

$$\int_0^{+\infty} e^{-tx}\varphi(t)\,dt \asymp \frac{\varphi(1/x)}{x}, \qquad 0 < x < +\infty. \tag{3.3}$$

2°. *If $\varphi(t) \nearrow$ but $(1-e^{-t})^{-\alpha}\varphi(t) \searrow$ for an $\alpha > 0$, then for enough small $v > 0$*

$$\int_{+0}^{+\infty} e^{-tv}\varphi(t)\,d[t] \asymp \frac{\varphi(1/v)}{v}. \tag{3.4}$$

3°. *If $\varphi(t) \searrow$ in $(0,+\infty)$ but $t^\delta\varphi(t) \nearrow$ or $(1-e^{-t})^\delta\varphi(t) \nearrow$ for a $\delta \in (0,1)$, then (3.3) and (3.4) correspondingly are true.*

3.2. The below two theorems are for the half-plane kernels.

Theorem 3.1. *Let $\omega(t) > 0$ be a non-decreasing, continuously differentiable function in $(0,+\infty)$, such that*

1°. *$\omega(+0) = 0$ and $\lim_{x\to+\infty} e^{-\varepsilon x}\omega(x) = 0$ for any $\varepsilon > 0$,*

2°. **(i)** *$\omega'(x) \nearrow$ but $x^{-\alpha}\omega'(x) \searrow$ for some $\alpha > 0$ or, alternatively,*

(ii) *$\omega'(x) \searrow$ but $x^\delta\omega'(x) \nearrow$ for some $\delta \in (0,1)$,*

Then

$$C_\omega(iy) \asymp \frac{1}{y^2\omega'(y)}, \qquad 0 < y < +\infty.$$

If along with 1° and 2° **(i)** *we have*

3°. *$\omega'(+\infty) = +\infty$ and $x^{-1}\omega'(x) \nearrow$ or $x^{-1}\omega'(x) \searrow$ but $x^{-\delta}\omega'(x) \nearrow$ for some $\delta \in (0,1)$, then there exists a constant M $(\equiv M_\omega) > 0$ for which*

$$|C_\omega(z)| \le \frac{M}{|z|^2\omega'(y)}, \qquad z = x + iy \in G^+.$$

The latter estimate is proved under the assumption that $\omega'(x) \nearrow$. Nevertheless, it is used for proving the following theorem which is for the case $\omega'(x) \searrow$.

Theorem 3.2. *Let $\omega(t) > 0$ be a continuously differentiable, non-decreasing function in $(0,+\infty)$, such that*

1°. *$x^{-1}\omega(x) \searrow$, $x^{-\delta}\omega(x) \nearrow$ for a $\delta \in (0,1)$ and $\omega'(x) \searrow$,*

2°. *$\omega(+\infty) = +\infty$ but $\lim_{x\to+\infty} e^{-\varepsilon x}\omega(x) = 0$ for any $\varepsilon > 0$.*

Then there exists a constant M $(\equiv M_\omega) > 0$ for which

$$|C'_\omega(z)| \le \frac{M}{|z|^2 y\omega'(y)}, \qquad z = x + iy \in G^+.$$

3.3. The following two theorems are true for the disc kernels

$$C_\omega(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k}, \quad |z| < 1 \qquad \Big(\Delta_0 = 1, \quad \Delta_k = -\int_0^1 x^k d\omega(x), \quad k = 1, 2, \dots\Big).$$

Theorem 3.3. *Let $\omega(t) > 0$ be a non-increasing, continuously differentiable function in $(0, 1)$, such that*

1°. $\omega(1-0) = 0$ *and* $\omega(+0) = 1$,

2°. **(i)** $|\omega'(x)| \searrow$ *but* $(1-x)^{-\alpha}|\omega'(x)| \nearrow$ *for an* $\alpha > 0$ *or, alternatively,*
(ii) $|\omega'(x)| \nearrow$ *but* $(1-x)^{\delta}|\omega'(x)| \searrow$ *for a* $\delta \in (0, 1)$.

Then

$$C_\omega(r) \asymp \frac{1}{(1-r)^2|\omega'(r)|}, \qquad 0 < r < 1.$$

If along with 1° *and* 2°**(i)**

3°. $\omega'(1-0) = 0$ *and* $(1-x)^{-1}|\omega'(x)| \searrow$ *or* $(1-x)^{-1}|\omega'(x)| \nearrow$ *but* $(1-x)^{-\delta}|\omega'(x)| \searrow$ *for a* $\delta \in (0, 1)$, *then there exists a constant* M $(\equiv M_\omega) > 0$ *such that*

$$|C_\omega(z)| \le \frac{M}{|1-z|^2|\omega'(|z|)|}, \qquad |z| < 1.$$

Theorem 3.4. *Let $\omega(t) > 0$ be a non-increasing, continuously differentiable function of $L_1(0, 1)$, such that $\omega(1-0) = 0$, $\omega(+0) = 1$ but $(1-x)^{-1}\omega(x) \nearrow$, $(1-x)^{-\delta}\omega(x) \searrow$ for a $\delta \in (0, 1)$ and $|\omega'(x)| \nearrow$ in $(0, 1)$. Then there exists a constant M $(\equiv M_\omega) > 0$ such that*

$$|C'_\omega(z)| \le \frac{M}{|1-z|^2(1-|z|)|\omega'(|z|)|}, \qquad |z| < 1.$$

Remark 3.1. The used evaluation method is applicable also if the condition of not more than power decrease is posed not on the derivative but immediately on the function $\omega(x)$. This leads to estimates similar to (3.2), where the power function in the dominator is linear and ω' is replaced by ω. However, we prefer the form (3.2) with ω' as it turns to be better for further application.

Remark 3.2. The ω-function-parameters for which the C_ω kernels are evaluated in Theorems 3.1 – 3.4 are strictly contained in the class of *functions of regular behavior*. Although, among them there are ω-functions for which

$$|\omega'(x)| \asymp (1-x)^\alpha \log^\beta \frac{1}{1-x} \quad \text{as} \quad x \to 1-0 \quad \text{and} \quad |\omega'(x)| \asymp x^\alpha \log^\beta \frac{1}{x} \quad \text{as} \quad x \to +0$$

for any given $\alpha \in (-1, +\infty)$ and $\beta > 0$.

3.4. The following two theorems are true for ω-kernels in the half-plane in the case of exponentially decreasing parameter–functions.

Theorem 3.5. *Let*

$$\omega(t) \equiv \omega_{\rho,\alpha}(t) = \int_0^t e^{-\rho/\sigma}\sigma^\alpha d\sigma, \qquad 0 < t < +\infty,$$

where $\rho > 0$ and α are any fixed real numbers. Then the kernel (3.1) *admits the following estimate:*

$$C_\omega(iy) \asymp \frac{1}{y^3\omega'(y)} \times \begin{cases} 1+y & \text{if } \alpha > -1, \\ (1+y)^{2+\alpha} & \text{if } \alpha < -1, \\ (1+y)\log^{-1}(e+y) & \text{if } \alpha = -1, \end{cases} \quad 0 < y < +\infty.$$

And if $\alpha > 0$, then there exists a constant $M \equiv M_{\rho,\alpha} > 0$ such that

$$|C_\omega(z)| \le M\frac{1+y}{|z|^2 y\omega'(y)}, \qquad z = x+iy \in G^+.$$

3.5. Some similar estimates are true for the kernels $C_\omega(z)$ in the unit disk, for the scale of parameter-functions

$$\omega(x) \equiv \omega_{\rho,\alpha}(x) = -K\int_x^1 \exp\left\{-\frac{\rho}{1-t}\right\}(1-t)^\alpha dt, \quad 0 \le x \le 1,$$

where $K = \left(\int_0^1 \exp\left\{-\frac{\rho}{1-t}\right\}(1-t)^\alpha dt\right)^{-1}$ and $\rho, \alpha > 0$ are any numbers.

Theorem 3.6. *For any real α and $\rho > 0$*

$$C_\omega(r) \asymp \frac{1}{(1-r)^3|\omega'(r)|}, \qquad 0 < r < 1.$$

And if $\alpha > 0$, then there exists a constant $M \equiv M_{\rho,\alpha} > 0$ such that

$$|C_\omega(z)| \le \frac{M}{|1-z|^2(1-|z|)|\omega'(|z|)|}, \qquad |z| < 1.$$

4. Projection theorems and the conjugate space of A^p_ω

4.1. While formula (1.3) of Theorem 1.2 describes the orthogonal projections of $L^2[0,2\pi]$ onto A^2_ω, our next theorem shows that the representation formula (1.2) defines an orthogonal projection of L^2_ω onto A^2_ω. Note that for $\omega(x) = 1-x$ one can find the proof of this statement (i.e. the projection $L^2_0 \longrightarrow A^2_0$) in [5, 8], and for $\omega(x) = (1-x)^{\alpha+1}$ $(\alpha > -1)$ (i.e. for $L^2_\alpha \longrightarrow A^2_\alpha$) the proof is first given in [2] (Theorem VII).

Before proving our theorem, note that if $\omega \in \Omega_A$ is a nonincreasing function, then one can easily show that

$$e_n(z) = \frac{z^n}{\sqrt{\Delta_n(\omega)}}, \qquad n = 0,1,2,\dots \tag{4.1}$$

is an orthonormal set in A^2_ω. As the polynomials are dense in A^2_ω, the set (4.1) is an orthonormal basis in A^2_ω. Besides, one can prove that if $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$ are any functions from A^2_ω, then

$$\|f\|^2_{2,\omega} = \sum_{n=0}^\infty |a_n|^2\Delta_n(\omega) \qquad \text{and} \qquad (f,g)_\omega = \sum_{n=0}^\infty a_n\bar{b}_n\Delta_n(\omega),$$

where the inner product $(f,g)_\omega$ is that induced from L^2_ω.

Theorem 4.1. *If $\omega \in \Omega_A$ is a nonincreasing function such that $\bigvee_0^1 \omega = 1$, then the following formula*

$$P_\omega f(z) = \frac{1}{2\pi}\iint_{|\zeta|<1} f(\zeta)C_\omega(z\bar{\zeta})d\mu_\omega(\zeta), \qquad |z| < 1, \quad f \in L^2_\omega,$$

is true for the orthogonal projection of L^2_ω *onto* A^2_ω.

Remark 4.1. While the sum of classes A^2_ω considered in Theorem 4.1 coincides with the whole set of functions holomorphic in $|z| < 1$ (Proposition 1.1), the sum of classes L^2_ω which are orthogonally projected to A^2_ω in Theorem 4.1coincides with the set of all measurable in $|z| < 1$ functions $F(z)$ for which $F(\rho_n e^{i\vartheta}) \in L^2[0, 2\pi]$ on a sequence $\rho_n = \rho_n(F) \uparrow 1$.

4.2. For proving some projection theorems from L^p_ω to A^p_ω $(1 \le p < \infty)$, which contain those known for $L^p_\alpha \longrightarrow A^p_\alpha$ [17, 18] in the particular case $\omega(x) = (1-x)^{\alpha+1}$ $(-1 < \alpha < \infty)$, the asymptotic estimates of Theorems 3.1 and 3.2 are used.

Definition 4.1. *A function* $\omega(x)$ *is of the class* $\widetilde{\Omega}_A$ *if along with* $\omega(x) \in \Omega_A$ *the below requirements* (A) *or* (B) *are satisfied.*

(A) $\omega(x) > 0$ *is nonincreasing, continuously differentiable in* $[0, 1)$ *and such that* $\omega'(x) \ne 0$ $(0 \le x < 1)$ *and*

- 1^o. $|\omega'(x)| \searrow$ *but* $(1-x)^{-\alpha}|\omega'(x)| \nearrow$ *for an* $\alpha > 0$,
- 2^o. $(1-x)^{-1}|\omega'(x)| \searrow$ *or alternatively* $(1-x)^{-1}|\omega'(x)| \nearrow$ *but* $(1-x)^{-\delta}|\omega'(x)| \searrow$ *for a* $\delta \in (0, 1)$.

(B) $\omega(x) > 0$ *is a nonincreasing, continuously differentiable function in* $[0, 1)$, *such that*

- 1^o. $\omega(1-0) = 0$, $\omega(x) \in L^1[0, 1)$,
- 2^o. $|\omega'(x)| \nearrow$ *but* $(1-x)^{\delta_1}|\omega'(x)| \searrow$ *for a* $\delta_1 \in (0, 1)$ *and* $(1-x)^{-1}|\omega(x)| \nearrow$ *but* $(1-x)^{-\delta_2}|\omega(x)| \searrow$ *for a* $\delta_2 \in (0, 1)$.

One can observe that if $\omega(x) \in \widetilde{\Omega}_A$, then $|\omega'(x)|$ is *of regular behavior* in $(0, 1)$. Indeed, if a function $\lambda(x)$ is such that $\lambda(x)x^{-\alpha} \nearrow$ and $\lambda(x)x^{-\alpha-\varepsilon} \searrow$ in $(0, 1)$, then for any $0 < \delta_0 < \delta < 1$

$$0 < \delta_0^{\alpha+\varepsilon} \le \delta^{\alpha+\varepsilon} \le \frac{\lambda(\delta x)}{\lambda(x)x^{-\alpha-\varepsilon}} x^{-\alpha-\varepsilon} = \frac{\lambda(\delta x)}{\lambda(x)} = \frac{\lambda(\delta x)(\delta x)^{-\alpha}}{\lambda(x)}(\delta x)^{\alpha} \le \delta^{\alpha} < 1, \quad 0 < x < 1.$$

The conditions of the following theorem in a sense are similar to those in a projection theorem proved by A.L.Shields and D.L.Williams [31].

Theorem 4.2. *Let* $\omega_1 \in \Omega_A$ *be continuously differentiable in* $[0, 1)$ *and let* $\omega_2 \in \widetilde{\Omega}_A$.

If $(1-x)^{-\beta}|\omega_1'(x)| \searrow$ *for some* $\beta > -1$ *and*

$$\left|\frac{\omega_1'(x)}{\omega_2'(x)}\right|(1-x)^{\Delta_1} \nearrow \qquad \text{and} \qquad \left|\frac{\omega_1'(x)}{\omega_2'(x)}\right|(1-x)^{\Delta_2} \searrow$$

for some Δ_1 *and* Δ_2 $(0 < \Delta_1 \le \Delta_2 < +\infty)$, *then the operator*

$$P_{\omega_2}F(z) = \frac{1}{2\pi}\iint_{|\zeta|<1} F(\zeta)C_{\omega_2}(z\bar{\zeta})d\mu_{\omega_2}(\zeta), \quad |z| < 1, \tag{4.2}$$

is a bounded projection of $L^1_{\omega_1}$ *onto* $A^1_{\omega_1}$.

In contrast to Theorem 4.2 which preassumes $\omega_1 \not\equiv \omega_2$ for $p = 1$, the next theorem shows that for $p > 1$ the representation formula of A^p_ω itself defines a bounded projection of L^p_ω onto A^p_ω.

Theorem 4.3. 1°. *Let both* $\omega_{1,2}(x) \in \Omega_A$ *be continuously differentiable in* $[0, 1)$ *and such that*

$$\left|\omega_{1,2}'(x)\right|(1-x)^{-\alpha_{1,2}} \nearrow \quad \text{and} \quad \left|\omega_{1,2}'(x)\right|(1-x)^{-\beta_{1,2}} \searrow$$

for some $-1 < \beta_1 \le \alpha_1$ *and* $-1 < \beta_2 \le \alpha_2$*. If* $\omega_2(x) \in \widetilde{\Omega}_A$ *and* $\alpha_1 + 1 < p(1+\beta_2)$*, then the operator* P_{ω_2} *defined by* (4.2) *is a bounded projection of* $L^p_{\omega_1}$ *onto* $A^p_{\omega_1}$ $(1 < p < +\infty)$.

2°. *If the functions* $\omega_{1,2}(x) \in \Omega_A$ *are continuously differentiable, nonincreasing in* $[0,1)$ *and* (4.2) *defines a bounded operator in* $L^p_{\omega_1}$ $(1 < p < +\infty)$*, then*

$$\int_0^1 \left| \frac{\omega_2'(x)}{\omega_1'(x)} \right|^q |\omega_1'(x)| dx < +\infty \qquad (1/p + 1/q = 1).$$

Remark 4.2. Obviously, the last condition is equivalent to $\alpha_1 + 1 < p(1+\alpha_2)$ in the case when $\omega_{1,2}(x) = (1-x)^{1+\alpha_{1,2}}$ $(\alpha_{1,2} = \beta_{1,2})$. Thus, for this particular case the previous Theorem 4.3 states that: *the operator (4.2) is a bounded projection of* $L^p_{\alpha_1}$ *onto* $A^p_{\alpha_1}$ *if and only if* $\alpha_1 + 1 < p(1+\alpha_2)$, what coincides with the known statement ([18], p. 12, Theorem 1.10) on projection of $L^p_{\alpha_1}$ onto $A^p_{\alpha_1}$. Some results on continuity of projections with $\omega(x) = (1-x)^{1+\alpha}$ in harmonic A^p_ω spaces having more general weights then a degree of $1-x$ are contained in [32].

Remark 4.3. *The requirements of Theorem 4.3 particularly are satisfied if*

$$|\omega'(x)| = M(1-x)^\alpha \log^\gamma \frac{a}{1-x}, \quad 0 \le x < 1, \qquad M = \left(\int_0^1 (1-t)^\alpha \log^\gamma \frac{a}{1-t} \right)^{-1},$$

where $\alpha,\gamma > 0$ *are any numbers and* $a \ge e^{\gamma/(\alpha-\beta)}$ *for some choice of* β.

4.3. Theorem 4.3 is used to establish

Theorem 4.4. *Let* $1 < p < +\infty$ *and* $1/q = 1 - 1/p$ *be any numbers. If* $\omega(x) \in \widetilde{\Omega}_A$ *satisfies*

$$\left|\omega'(x)(1-x)^{-\alpha}\right| \nearrow \quad \text{and} \quad \left|\omega'(x)(1-x)^{-\beta}\right| \searrow \qquad (0 \le x < 1)$$

for some $-1 < \beta \le \alpha$ *such that* $1+\alpha < q(1+\beta)$*, then the set of bounded linear functionals in* A^p_ω *is completely described by the formula*

$$\Phi(f) = (f,g)_\omega = \frac{1}{2\pi} \iint_{|\zeta|<1} f(\zeta)\, \overline{g(\zeta)}\, d\mu_\omega(\zeta), \quad f \in A^p_\omega, \quad g \in A^q_\omega, \tag{4.3}$$

i.e. $(A^p_\omega)^* = A^q_\omega$ $(1/p + 1/q = 1)$ *in the sense of isomorphism.*

Remark 4.4. Another form of linear functionals in A^p_ω, which differs from (4.3), is obtained in [21]. This work is based on the representations of some spaces A^p_α which contain the spaces A^p_ω considered in [21].

5. Weighted classes of δ-subharmonic functions

5.1. A function $u(z)$ is said to be δ–subharmonic in a domain G and ν is said to be its associated measure if

$$u(z) = u_1(z) - u_2(z) \quad \text{and} \quad \nu = \nu_1 - \nu_2,$$

where $u_{1,2}(z)$ are subharmonic functions in G, possessing Riesz associated measures $\nu_{1,2}$. We say that two δ–subharmonic functions

$$u(z) = u_1(z) - u_2(z) \quad \text{and} \quad v(z) = v_1(z) - v_2(z), \quad z \in G,$$

are equal, i.e. $u(z) = v(z)$, if everywhere in G

$$u_1(z) + v_2(z) = u_2(z) + v_1(z).$$

Besides, we assume that the associated measure ν of the δ–subharmonic function $u(z)$ is minimally decomposed in the Jordan sense (see, for instance [33], Ch. III, Sec. 4, 11 Corollary), i.e.

$$\nu = \nu_+ - \nu_-, \qquad (\text{supp }\ \nu_+)\bigcap(\text{supp }\ \nu_-) = \not{o},$$

where $\nu_\pm$ are positive and negative variations of ν. Also we assume that $0 \notin \overline{\text{supp }\ \nu}$ (i.e. $d_0 = \inf\{|\lambda| : \lambda \in \text{supp}\nu\} > 0$) and use the following generalization of Nevanlinna's characteristic:

$$T(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\vartheta})\,d\vartheta + \int_0^r \frac{n_-(t)}{t}dt, \qquad n_-(t) = \iint_{|\zeta|<t} d\nu_-(\zeta).$$

Along with the equilibrium relation

$$u(0) + T(r,-u) = T(r,u), \quad 0 < r < R,$$

the characteristic $T(r,u)$ arises from the following similarity of the Jensen–Nevanlinna formula for functions δ–subharmonic in $|z| < R \le +\infty$:

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta})\,d\vartheta - \iint_{|\zeta|<r} \log\frac{r}{|\zeta|}d\nu(\zeta), \quad 0 < r < R.$$

In its turn, this formula holds by taking $z = 0$ in the Riesz representation of $u(z)$ in $|z| < r < R$ (i.e. in the difference of the Riesz representations of u_1 and u_2 in $|z| < r$):

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\vartheta} - z|^2}u(re^{i\vartheta})d\vartheta + \iint_{|\zeta|<1} \log\left|\frac{r(\lambda - z)}{r^2 - \overline{\lambda}z}\right| d\nu(\lambda), \quad |z| < r.$$

Henceforth we consider only functions $\omega(x) \in \Omega_N$, what means the additional to $\omega(x) \in \Omega_A$ requirement $\omega(1) = \omega(1-0) = 0$, and we assume that also $\omega(x)$ is decomposed in its positive and negative variations, i.e. for all $x \in [0,1]$

$$\omega(x) = \omega_+(x) - \omega_-(x), \quad \omega_\pm(x) = \int_x^1 (d\omega(t))^\pm, \quad \bigvee_x^1 \omega = \bigvee_x^1 \omega_+ + \bigvee_x^1 \omega_- = \omega_+(x) + \omega_-(x).$$

Definition 5.1. *We call N_ω° the set of those functions $u(z)$ δ–subharmonic in $|z| < 1$, for which*

$$\int_0^1 T(r,u)|d\omega(r^2)| < +\infty.$$

Remark 5.1. The sum $\bigcup_{\omega\in\Omega_N} N_\omega^\circ$ coincides with the set of all functions δ–subharmonic in $|z| < 1$. The subset of N_ω° consisting of functions of the form $u(z) = \log|f(z)|$, where $f(z)$ is meromorphic in $|z| < 1$, coincides with Nevanlinna's weighted class (1) when $\omega(x) = (1-x)^{1+\alpha}$ $(-1 < \alpha < +\infty)$.

5.2. Theorem 5.1. 1°. *For any $\omega(x) \in \Omega_N$ and any fixed point $|\zeta| < 1$ the function*

$$b_\omega(z,\lambda) = \exp\left\{-\int_{|\lambda|^2}^1 C_\omega\left(\frac{zt}{\lambda}\right)\frac{\omega(t)}{t}dt\right\}, \quad |z| < |\lambda|,$$

possesses analytic continuation to the whole $|z| < 1$, where it has unique, simple zero at $z = \lambda$.

2°. *Let $\omega(x) \in \Omega_N$ and let $\nu \geq 0$ be any measure which is bounded inside $|z| < 1$ and such that $0 \notin \overline{\text{supp } \nu}$ and*

$$\iint_{|\lambda|<1} \left(\int_{|\lambda|^2}^{1} |\omega(t)| dt \right) d\nu(\lambda) < +\infty.$$

Then the Green type potential

$$J_\omega(z) = \iint_{|\lambda|<1} \log |b_\omega(z, \lambda)| d\nu(\lambda)$$

presents a subharmonic function in $|z| < 1$

Remark 5.2. The function $b_\omega(z, \zeta)$ is a generalization of the Blaschke-Djrbashian factor [1, 2], which is the case $\omega(x) = (1 - x)^{1+\alpha}$ $(-1 < \alpha < +\infty)$ of $b_\omega(z, \zeta)$. One has to note that M.M.Djrbashian's factors later were independently found by M.Tsuji [34] (Ch. IV) for $\omega(x) = (1 - x)^n$, $n = 1, 2, \ldots$. For the latest repetition of the same particular case of M.M.Djrbashian's factorization of 1945, see [35].

Theorem 5.2. *If $u(z) \in \widetilde{A}_\omega$ for an $\omega(x) \in \Omega_N$, then the associated measure ν of $u(z)$ satisfies the condition*

$$\iint_{|\zeta|<1} \left(\int_{|\zeta|^2}^{1} |\omega(t)| dt \right) d\nu_\pm(\zeta) = \iint_{|\zeta|<1} \left(\int_{|\zeta|^2}^{1} [\omega_+(t) + \omega_-(t)] dt \right) d\nu_\pm(\zeta) < +\infty, \quad (5.1)$$

and the following Riesz type representation is true in $|z| < 1$:

$$u(z) = -u(0) + \iint_{|\zeta|<1} \log |b_\omega(z, \zeta)| d\nu(\zeta) + \frac{1}{\pi} \iint_{|\zeta|<1} \text{Re} \left\{ C_\omega(z\overline{\zeta}) \right\} u(\zeta) \, d\mu_\omega(\zeta). \quad (5.2)$$

Remark 5.3. In the particular case $\omega(x) = (1 - x)^\alpha$ $(-1 < \alpha < +\infty)$ and $u(z) = \log |f(z)|$, where $f(z)$ is a meromorphic function in $|z| < 1$, the representation (5.2) becomes M.M.Djrbashian's canonical factorization [1, 2]. Besides, (5.1) becomes the well known condition

$$\sum_k (1 - |z_k|)^{2+\alpha} < +\infty$$

established by R.Nevanlinna for zeros and poles of functions $f(z)$ from his weighted class (1) (see [3], Sec 216).

Remark 5.4. One can see that $N_\omega^\circ \subset N_{\omega_1} \subseteq N_\omega$ $(\omega(x) = \omega(x^2))$, where N_ω is the class of those functions δ–subharmonic in $|z| < 1$, for which $L_\omega u(z)$ belongs to Nevanlinna's class, i.e. $L_\omega u(z)$ is representable in $|z| < 1$ as the difference of two nonpositive subharmonic functions. The descriptive representations of N_ω classes are established in [16]. Using a representation similar to (1.5) (with $\omega_1 \equiv \omega$), one can prove the representation of [16] with an absolutely continuous measure in the ω–parametered Poisson integral, i.e. to a similarity of (5.2) with somewhat different Green type potential and a Poisson type integral over $|z| = 1$ with an absolutely continuous measure.

6. Representations in the whole complex plane

6.1. We shall deal with the spaces $A_\omega^p(\mathbb{C})$ $(1 \leq p < +\infty)$ of entire functions satisfying the condition

$$\|F\|_{p,\omega} = \left\{ \frac{1}{2\pi} \iint_{\mathbb{C}} |F(\zeta)|^p |d\mu_\omega(\zeta)| \right\}^{1/p} < +\infty, \quad (6.1)$$

where $d\mu_\omega(\rho e^{i\vartheta}) = -d\omega(\rho^2)d\vartheta$ and $\omega \in \Omega_A^\infty$, i.e. $\omega(x)$ is a strictly decreasing function on the whole half-axis $[0, +\infty)$, such that $\omega(0) = 1$ and

$$\Delta_n^\infty(\omega) = -\int_0^{+\infty} t^n d\omega(t) < +\infty \quad \text{for any} \quad n = 0, 1, 2 \dots$$

We denote $L_\omega^p(\mathbb{C})$ the corresponding Lebesgue spaces.

One can show that A_ω^p $(1 \le p < +\infty)$ *is a Banach space with the norm (6.1).* On the other hand, the following assertion is true.

Lemma 6.1. *For any $p \in [1, +\infty)$ the sum $\bigcup_{\omega \in \Omega_A^\infty} A_\omega^p(\mathbb{C})$ coincides with the set of all entire functions.*

6.2 Observing that under the above conditions

$$\lim_{n\to\infty} \sqrt[n]{|\Delta_n^R(\omega)|} = R^2 \quad \text{for} \quad \Delta_n^R(\omega) = -\int_0^{R^2} t^n |d\omega(t)| \quad \text{and} \quad \forall R \in (0, +\infty],$$

we start by the following representations.

Theorem 6.1. *Let $F(z) \in A_\omega^2(\mathbb{C})$, where $\omega \in \Omega_A^\infty$. Then for all $z \in \mathbb{C}$*

$$F(z) = \frac{1}{2\pi} \iint_{\mathbb{C}} F(\zeta) C_\omega^\infty(z\overline{\zeta}) d\mu_\omega(\zeta), \tag{6.2}$$

$$F(z) = -\overline{F(0)} + \frac{1}{\pi} \iint_{\mathbb{C}} \{\text{Re } F(\zeta)\} C_\omega^\infty(z\overline{\zeta}) d\mu_\omega(\zeta), \tag{6.2$'$}$$

where $C_\omega^\infty(z) = \sum_{n=0}^\infty z^n / \Delta_n^\infty(\omega)$ is M.M.Djrbashian's Cauchy type kernel [36].

Remark 6.1. One can verify that *the set (4.1)* (with $\Delta_n^\infty(\omega)$ instead $\Delta_n(\omega)$) *is an orthonormal basis in $A_\omega^2(\mathbb{C})$ $(\omega \in \Omega_A^\infty)$.* Besides, one can be convinced that the following similarity of Theorem 4.1 is true.

Theorem 6.2. *If $\omega \in \Omega_A^\infty$, then the following formula*

$$P_\omega f(z) = \frac{1}{2\pi} \iint_{\mathbb{C}} f(\zeta) C_\omega^\infty(z\overline{\zeta}) d\mu_\omega(\zeta), \qquad z \in \mathbb{C}, \quad f \in L_\omega^2(\mathbb{C}), \tag{6.3}$$

is true for the orthogonal projection of $L_\omega^2(\mathbb{C})$ onto $A_\omega^2(\mathbb{C})$.

6.3. In virtue of Theorem 6.2, it is evident that for any $\omega \in \Omega_A^\infty$ the representation (6.2),(6.3) with $f \in L_\omega^2(\mathbb{C})$ describes the whole class $A_\omega^2(\mathbb{C})$, i.e. $A_\omega^2(\mathbb{C})$ coincides with the set of all functions representable in that form. Besides, the following similarity of Theorem 1.2 is true.

Theorem 6.3. *Let $\omega \in \Omega_A^\infty$ be continuously differentiable in $[0, +\infty)$ and such that $\omega(+\infty) = 0$, $\omega'(x) < 0$ and is bounded on $[0, +\infty)$ and $0 > \int_0^{+\infty} t^{-1} d\omega(t) > -\infty$. Then the function*

$$\widetilde{\omega}(x) = -\int_0^{+\infty} \omega\Big(\frac{x}{t}\Big) d\omega(t), \qquad 0 < x < +\infty,$$

belongs to Ω_A^∞ and $A_{\widetilde{\omega}}^2(\mathbb{C})$ coincides with the set of all functions representable in the form

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\vartheta}) C_\omega^\infty(ze^{-i\vartheta}) d\vartheta, \qquad z \in \mathbb{C}, \quad \varphi(e^{i\vartheta}) \in L^2[0, 2\pi]. \tag{6.4}$$

For any $F(z) \in A^2_{\widetilde{\omega}}(\mathbb{C})$ there exists a unique function $\varphi_0(z)$ of the ordinary Hardy space H^2, such that (6.4) is true with $\varphi_0(e^{i\vartheta})$. This function is found by the formula

$$\varphi_0(z) = L^{\infty}_{\omega} F(z) = -\int_0^{+\infty} F(tz) d\omega(t), \qquad |z| < 1.$$

Besides, $\|\varphi_0\|_{H^2} = \|F\|_{2,\widetilde{\omega}}$ and $\varphi - \varphi_0 \perp H^2$ for any $\varphi(e^{i\vartheta}) \in L^2[0, 2\pi]$ for which (6.4) is true. The operator L_ω is an isometry $A^2_{\widetilde{\omega}}(\mathbb{C}) \longrightarrow H^2$, and the integral (6.4) defines $(L_\omega)^{-1}$ on H^2.

Remark 6.2. *The sum of the spaces $A^2_{\widetilde{\omega}}(\mathbb{C})$ considered in Theorem 6.3 coincides with the set of all entire functions.*

6.4. For functions of $A^p_\omega(\mathbb{C})$ the similarities of Shamoian's representation is true. The following theorem is proved.

Theorem 6.4. *Let $1 \le p < +\infty$ and let $\omega(x) \in \Omega^\infty_A$ and $\bigvee_0^\infty \omega = 1$. Then:*

1°. *Any function $F(z) \in A^p_\omega(\mathbb{C})$ is representable in the form*

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} C_{\omega_1}(z e^{-i\vartheta}) \varphi(e^{i\vartheta}) d\vartheta, \quad |z| < 1, \tag{6.5}$$

where $\omega_1(x) = \omega(x^2)$ and $\varphi(z) = L_{\omega_1} F(z) \in H^p$.

2°. *In A^p_ω we have $\|L_{\omega_1}\| \le 1$, and* (6.5) *represents $L^{-1}_{\omega_1}$ in $L_{\omega_1} A^p_\omega(\mathbb{C})$.*

6.4. The assertions of Theorem 6.3 remain valid for the case

$$\omega'(x) = -C_0\, e^{-\gamma x^\rho} x^{\mu\rho - 1}, \quad 0 < x < +\infty, \quad C_0 = \left[\int_0^{+\infty} e^{-\gamma x^\rho} x^{\mu\rho-1} dx\right]^{-1},$$

where $\gamma, \rho, \mu > 0$ are any numbers, though the requirements on $\widetilde{\omega}'(x)$ providing the inclusion $\omega(x) \in \Omega^\infty_A$ in Theorem 6.3 are not satisfied. However, the following improvement of M.M.Djrbashian's [2] Theorem XIV_1 is true.

Theorem 6.5. *The Hilbert space of entire functions which satisfy the condition*

$$\|F\| = \left\{\frac{1}{2\pi} \int_0^{+\infty} d\vartheta \int_0^{2\pi} |F(re^{i\vartheta})|^2 e^{-2\gamma r^\rho} r^{2\mu\rho - \rho/2 - 1} dr\right\}^{1/2} < +\infty, \tag{6.6}$$

where $\gamma, \rho, \mu > 0$ are any numbers, coincides with the set of all functions representable in the form

$$F(z) = \frac{\rho \gamma^\mu}{2\pi} \int_0^{2\pi} E_\rho\left(\gamma^{1/\rho} z e^{-i\vartheta}, \mu\right) \varphi(e^{-i\vartheta}) d\vartheta, \quad z \in \mathbb{C}, \quad \varphi(e^{-i\vartheta}) \in L^2[0, 2\pi], \tag{6.15}$$

where $E\rho(z, \mu) = \sum_{k=0}^\infty z^k \left[\Gamma\left(\mu + k/\rho\right)\right]^{-1}$ is the well-known Mittag–Leffler type function.

For any $F(z)$ of the space (6.6) there exists a unique function $\varphi_0(z) \in H^2(|z| < 1)$, such that (6.15) is true with $\varphi_0(e^{i\vartheta})$. This function can be found by the formula

$$\varphi_0(z) = \int_0^{+\infty} F(tz) e^{-\gamma t^\rho} t^{\mu\rho - 1} dt, \quad |z| < 1. \tag{6.17}$$

Besides, $\|\varphi_0\|_{H^2} = \|F\|$ and $\varphi - \varphi_0 \perp H^2$ for any $\varphi(e^{-i\vartheta}) \in L^2[0, 2\pi]$ for which (6.15) is true.

7. Biorthogonal systems of functions in A^2_ω

The isometries between H^2 $(|z|<1)$ and the spaces $A^2_{\widetilde{\omega}}$ and $A^2_{\widetilde{\omega}}(\mathbb{C})$ established in Theorems 1.2, 6.3 and 6.5 permit to convert the known facts in H^2 into similar statements in A^2_ω. Particularly, the results of M.M.Djrbashian–H.M.Hayrapetyan [37, 38, 39] *on biorthogonal systems of functions in* H^2 are convertible into similar statements in $A^2_{\widetilde{\omega}}$ and $A^2_{\widetilde{\omega}}(\mathbb{C})$. The below propositions contain the results which hold in $A^2_{\widetilde{\omega}}(\mathbb{C})$.

For simplicity we consider the case when the knots are of multiplicity 1, i.e. everywhere below we assume that $\{\alpha_j\}_1^\infty$ is a sequence of *pairwise different* numbers in $|z|<1$, which satisfies the Blaschke condition

$$\sum_{j=1}^{\infty}(1-|\alpha_j|)<+\infty. \tag{7.1}$$

It is said that $\{\alpha_j\}_1^\infty\in\Delta$ if the sequence $\{\alpha_j\}_1^\infty$ is uniformly separated, i.e.

$$\inf_{k\geq 1}\prod_{j=1,\ j\neq k}\left|\frac{\alpha_j-\alpha_k}{1-\overline{\alpha_j}\alpha_k}\right|=\delta>0.$$

Also, we introduce the Blaschke product with zeros at $\{\alpha_j\}_1^\infty$:

$$B(z)=\prod_{j=1}^{\infty}\frac{\alpha_j-z}{1-\overline{\alpha_j}z}\frac{|\alpha_j|}{\alpha_j}.$$

Assuming that henceforth the functions $\widetilde{\omega}(x)$ and $\omega(x)$ are those of Theorem 6.3 or Theorem 6.5, we denote the isometry $H^2\longrightarrow A^2_{\widetilde{\omega}}(\mathbb{C})$ of that theorems as

$$T^\infty_\omega(f(z))=\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\vartheta})C^\infty_\omega(ze^{i\vartheta})d\vartheta,\quad |z|<1,\quad (T_\omega=(L^\infty_\omega)^{-1}).$$

A well known inequality due to H.Shapiro and A.Shields [40] takes the following form:

Proposition 7.1. *If* $\{\alpha_k\}_1^\infty\in\Delta$, *then for any* $F(z)\in A^2_{\widetilde{\omega}}(\mathbb{C})$

$$\sum_{j=1}^{\infty}(1-|\alpha_j|^2)|L^\infty_\omega F(\alpha_j)|^2\leq C\|F\|_{\tilde{\omega},2},$$

where C *is a constant.*

For giving a series of propositions which mainly follow the account of [39], first denote

$$r_k(z)=\frac{1}{1-\overline{\alpha_k}z}\quad\text{and}\quad\Omega_k(z)=\frac{B(z)}{z-\alpha_k}.$$

One can verify that the entire functions $T^\infty_\omega(r_k(z))=r^\infty_{k,\omega}(z)$ and $T^\infty_\omega(\Omega_k(z))=\Omega^\infty_{k,\omega}(z)$ admit the following representations:

$$r^\infty_{k,\omega}(z)=C^\infty_\omega(\overline{\alpha_k}z)\quad\text{and}\quad\Omega^\infty_{k,\omega}(z)=\lim_{\rho\to 1-0}\sum_{n=0}^{\infty}\frac{z^n\rho^n}{\Delta_n(\omega)}\sum_{m=0}^{\infty}\alpha_k^m\rho^{m+1}b_{n+m+1}, \tag{7.2}$$

where b_n are the coefficients of the power expansion $B(z)=\sum_{n=0}^\infty b_nz^n$ $(|z|<1)$.

Proposition 7.2. *If the sequence* $\{\alpha_k\}_1^\infty$ *does not satisfy the Blaschke condition, i.e. the series* (7.1) *is divergent, then both systems are complete in* $A^2_{\tilde{\omega}}(\mathbb{C})$.

Further, we introduce $\lambda^\infty_{2,\omega}\{\alpha_k\}$ as the set of all functions $F(z) \in A^2_{\tilde{\omega}}(\mathbb{C})$ for which there exist $\psi(z) \in H^2$, $\psi(0) = 0$ such that the boundary values of $\psi(z)/B(1/z)$ coincide with those of $L^\infty_\omega F(z)$ almost everywhere on $|z| = 1$.

Proposition 7.3. *The functions of* (7.2) *belong to* $\lambda^\infty_{2,\omega}\{\alpha_k\}$*, and the system* (7.2) *is biorthogonal in* $A^2_{\tilde{\omega}}(\mathbb{C})$*, i.e*

$$(r^\infty_{k,\omega}, \Omega^\infty_{k,\omega})_{\tilde{\omega}} \equiv \iint_{\mathbb{C}} r^\infty_{k,\omega}(\zeta)\overline{\Omega^\infty_{k,\omega}(\zeta)}d\mu_{\tilde{\omega}}(\zeta) = \begin{cases} 1, & if \quad \nu = k \\ 0, & if \quad \nu \neq k. \end{cases}$$

Proposition 7.4. *Let* $F(z) \in A^2_{\tilde{\omega}}(\mathbb{C})$*. Then* $F(z) \in \lambda^\infty_{2,\omega}\{\alpha_k\}$ *if and only if*

$$\int_{|\zeta|=1} \frac{L^\infty_\omega F(\zeta)}{B(\zeta)} \frac{d\zeta}{\zeta - z} \equiv 0, \quad z \in \mathbb{C}.$$

Proposition 7.5. *Any function* $f(z) \in A^2_{\tilde{\omega}}(\mathbb{C})$ *is representable in the form*

$$f(z) = f_1(z) + f_2(z), \quad \|f\|^2 = \|f_1\|^2 + \|f_2\|^2,$$

where $f_1(z) \in \lambda^\infty_{2,\omega}\{\alpha_k\}$ *and*

$$f_2(z) = T_\omega(B(z)g(z)) \in A^2_{\tilde{\omega}}(\mathbb{C}), \quad g(z) = \frac{1}{2\pi} \int_{|\zeta|=1} \frac{L^\infty_\omega f(\zeta)}{B(\zeta)} \frac{d\zeta}{\zeta - z} \in H^2.$$

Proposition 7.6. *If* $\{\alpha_k\}_1^\infty \in \Delta$ *and* $\{w_k\}_1^\infty$ *is any sequense for which*

$$\sum_{k=1}^\infty (1 - |\alpha_k|)|w_k|^2 < +\infty,$$

then there exists unique function $F(z) \in \lambda_2^{\infty,\omega}\{\alpha_k\}$ *such that*

$$L^\infty_\omega F(\alpha_k) = w_k, \qquad k = 1, 2, \ldots \tag{7.3}$$

This function is representable in the form

$$F(z) = \sum_{k=1}^\infty w_k \Omega^\infty_{k,\omega}(z), \quad z \in \mathbb{C},$$

where the series converges in the norm of A^2_ω*, and* $F(z)$ *is the solution of the interpolation problem* (7.3) *with minimal norm.*

Proposition 7.7. *Each of the sets* $\{(1-|\alpha_k|^2)^{1/2} r^\infty_{k,\omega}(z)\}_1^\infty$ *and* $\{(1-|\alpha_k|^2)^{-1/2}\Omega^\infty_{k,\omega}(z)\}_1^\infty$ *is a nonconditional basis in* $\lambda^\infty_{2,\omega}\{\alpha_k\}$ *if and only if* $\{\alpha_k\}_1^\infty \in \Delta$.

Proposition 7.8. *If* $\{\alpha_k\}_1^\infty \in \Delta$*, then any function* $F(z) \in \lambda^\infty_{2,\omega}\{\alpha_k\}$ *is representable by both series*

$$F(z) = \sum_{k=1}^\infty c_k(F) r^\infty_{k,\omega}(z) = \sum_{k=1}^\infty L^\infty_\omega F(\alpha_k)\Omega^\infty_{k,\omega}(z), \quad z \in \mathbb{C}, \quad c_k(F) = (F, \Omega^\infty_{k,\omega})_{\tilde{\omega}}$$

which are convergent in the norm of $A^2_{\tilde{\omega}}(\mathbb{C})$.

Remark. The biorthogonal systems of this section qualitatively differ from those investigated by A.F.Leontev [41].

REFERENCES

1. *M. M. Djrbashian,* "On canonical representation of functions meromorphic in the unit disc" [in Russian], DAN of Armenia, 1945, Vol. 3, No. 1, pp. 3 – 9.
2. *M. M. Djrbashian,* "On the representability problem of analytic functions" [in Russian], Soobsch. Inst. Math. and Mech. AN of Armenia, 1948, Vol. 2, pp. 3 – 40.
3. *R. Nevanlinna.* Eindeutige Analytische Funktionen. Springer, Berlin, 1936.
4. *L. Biberbach,* "Zur theorie und praxis der konformen abbildung", Palermo Rendiconti, 1914, Vol. 38, pp. 98 – 118.
5. *J. L. Walsh.* Interpolation and approximation by rational functions. Amer. Math. Soc. Publ., New York, 1935.
6. *T. Carleman,* "Űber die approximation analytischer funktionen durch lineare aggregate von vorgegebenen potenzen", Arkhiv för Mat., Astr. och Fys., 1922, Vol. 17, no. 9.
7. *S. Bergman,* "Uber unendliche Hermitische Formen, die zu einem Bereiche gehoren, nebst Anwendungen auf Fragen der Abbildung durch Funktionen von zwei komplexen Veranderlichen", Math. Z. 1929, Vol. 29, pp. 641 – 677.
8. *W. Wirtinger,* "Űber eine minimumaufgabe im gebiet der analytischen functionen", Motatshefte fűr Math. und Phys., 1932, Vol. 39, pp. 377 – 384.
9. *Gh. Hardy, J. E. Littlewood,* "Some properties of functional integrals II", Mathematische Zeitschrift, 1932, Vol. 34, pp. 403 – 439.
10. *M. V. Keldysch,* "Sur les conditions pour qu'un systeme de polynomes orthogonaux avec un poids soit ferme", Coptes Rendus (Doklady) Academie des Sciences USSR (ns), 1941, Vol. 30, pp. 778 – 780.
11. *M. M. Djrbashian.* Integral transforms and representations of functions in the complex domain [in Russian], Nauka, Moscow, 1966.
12. *M. M. Djrbashian,* "A generalized Riemann–Liouville operator and some of its applications", Math. USSR Izv., 1968, Vol. 2, pp. 1027 – 1065.
13. *M. M. Djrbashian,* "Theory of factorization of functions meromorphic in the unit disc", Math. USSR Sbornik, 1969, Vol. 8, No. 4, pp. 493 – 591.
14. *M. M. Djrbashian,* "Theory of factorization and boundary properties of functions meromorphic in the disc", Proceedings of the International Congress of mathematicians, Vancouver, 1974, pp. 197 – 202.
15. *M. M. Djrbashian and V. S. Zakarian.* Classes and boundary properties of functions meromorphic in the disc. [in Russian] Nauka, Moscow, 1993.
16. *A. M. Jerbashian,* "An extension of the factorization theory of M. M. Djrbashian", J. of Contemp. Math. Analysis, 1995, Vol. 30. No. 2, pp. 39 – 61.
17. *A. E. Djrbashian, F. A. Shamoian.* Topics in the theory of A^p_α spaces, Teubner Verlag., Leipzig, 1988.
18. *H. Hedenmalm, B. Korenblum, K. Zhu.* Theory of Bergman Spaces, Springer Verlag, 2000.
19. *G. M. Gubreev, A. M. Jerbashian,* "Functions of generalized bounded type in spectral theory of non-weak contractions", Journal of Operator Theory, 1991, Vol. 26, pp. 155 – 190.
20. *A. M. Jerbashian, K. L. Avetisyan,* "On the general theory of classes of regular functions weighted integrable over the unit disc" [in Russian], National Ac. of Sci. of Armenia, Reports, 2002, Vol. 102, no. 2, pp. 105 – 112
21. *F. A. Shamoyan,* "Some remarks on the parametric representation of Nevanlinna–Djrbashian classes", Math. Notes, 1993, Vol. 52, No. 1–2, pp. 727 – 737.

22. *F. A. Shamoyan,* “Parametrical representation and description of the root sets of classes of functions holomorphic in in the disc” [in Russian], Sib. Mat. J., 1999, Vol. 40, No. 6, pp. 1420 – 1440.
23. *F. A. Samoyan, E. N. Shubabko,* “Parametrical reprezentation of some classes of holomorphic function in the disc”, Operator Thory; Advances and Application, Vol. 113, pp. 332 – 339, Birkhauser Verlag, Basel, 2000.
24. *F. A. Shamoyan, E. N. Shubabko*, “On a class opf functions holomorphic in the disc”, Investigations on linear operators and function theory [in Russian], Vol. 29, pp. 242 – 256, POMI, 2001, St. Petersburg.
25. *F. A. Shamoian* “Diagonal mapping and representation questions in anisothropic spaces of functions holomorphic in the polydisc” [in Russian], Sib. Mat. J., 1990, Vol. 31, No. 3, pp. 197 –216.
26. *K. L. Avetisyan.* “Fractional integration and integral representations in weighted classes of harmonic functions”, Analysis Math., 2000, Vol. 26, No. 3, pp. 161 – 174.
27. *M. M. Djrbashian, V. S. Zakarian,* “Boundary properties of subclasses of meromorphic functions of bounded type”, Math. USSR Izv., 1970, Vol. 4, pp. 1273 – 1354.
28. *M. M. Djrbashian, V. S. Zakarian,* “Boundary properties of subclasses of meromorphic functions of bounded type”, Izv. Akad. Nauk Armenii, Math., 1971, Vol. 6, No. 2-3, pp. 182 – 194.
29. *A. M. Jerbashian,* “Evaluation of ω–kernels of M.M.Djrbashian”, Journal of Inequalities and Applications (to appear in 2003).
30. *A. H. Karapetyan,* Some questions of integral representations in multidimensional complex analysis [in Russian], Reference by the Author to Candidate Thesis, Yerevan, 1988.
31. *A. L. Shields, D. L. Williams,* “Bounded projections, duality, and multipliers in spaces of analytic functions”, Trans. Amer. Math. Soc., 1971, v. 162, pp. 287 – 302.
32. *O. Blasco, S. Perez-Esteva,* “L^p continuity of projectors of weighted harmonic Bergman spaces”, Collect. Math., 2000, v. 51, no. 1, pp 49 – 58.
33. *N. Dunford, T. Schwarz.* Linear operators, Part. I, Intersci. Pub., New York, London, 1958.
34. *M. Tsuji.* Potential theory in modern function theory, Maruzen Co. Ltd. Tokyo, 1975.
35. *Ch. Horowitz.* “Factorization of functions in generalized Nevanlinna classes”, Proc. Amer. Math. Soc., Vol. 127, No. 3, 1999, pp. 745 – 751.
36. *M. M. Djrbashian,* “Factorization of functions meromorphic in the finite plane”, [in Russian], Izv. Akad. Nauk Armenii, Math., 1970, Vol. 5, No. 6, pp. 453 – 485.
37. *M.M.Djrbashian,* “Biorthogonal systems of rational functions and representations of the Cauchy kernel” [in Russian], Izv. Akad. Nauk Armenii, Math., 1973, Vol. 8, No. 5, pp. 384 – 409.
38. *M.M.Djrbashian,* “Biorthogonal systems and solution of interpolation problem with knots of bounded multiplicity in H_2” [in Russian], Izv. Akad. Nauk Armenii, Math., 1974, Vol. 9, No. 5, pp. 339 – 373.
39. *H. M. Hayrapetyan,* “On basicity of some biorthogonal systems in the complex domain”, [in Russian], Izv. Akad. Nauk Armenii, Math., 1975, vol. 10, no. 2, pp. 133 – 152.
40. *H. Shapiro and A. Shields,* “On some interpolation problems for analytic functions”, Amer. J. Math., 1961, vol. 83, pp 513 – 532.
41. *A. F. Leontev.* Generalizations of exponential series, Moscow, Nauka, 1981.

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