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Abstract. In this paper we give a new proof of Riemann's well known mapping theorem. The suggested method permits to proof an analog of that theorem for three dimensional case.

Keywords: Quasi - conform mapping, Riemann's theorem.

1 Introduction

By Liouville's theorem, see [2], p. 130, in three dimensional case, only superposition of isometric, dilatation and inverse transformations are conformal. To get an analogy for Riemann's mapping theorem, one introduce a new family of mappings named quasi - conformal. This family is wider, nevertheless we have not a natural analogy of conformal mappings like of two dimensional case. In this paper we introduce a new family of mappings, named week - conformal. For this, new family of mappings, we have more natural generalization of Riemann's theorem.

The prove of the main result of this paper is interesting for two dimensional case too. Actually, we give a new prove of Riemann's classical theorem, where the specific properties of complex analysis do not used. This permits to find its three dimensional analog.

2 Classes of mappings

For an arbitrary matrix

$$\det(M) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

with eigenvalues $\lambda_1, \quad \dots, \lambda_n$ let us denote by

$$|M|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} \lambda_i^2,$$

$$\mathrm{tr}(M) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

and

$$\det(M) = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \prod_{i=1}^{n} \lambda_i .$$

Let $\varphi(x,y,z) = (A,B,C)$ be continuous differentiable mapping. Let us denote by

$$J = \begin{pmatrix} A'_x & A'_y & A'_z \\ B'_x & B'_y & B'_z \\ C'_x & C'_y & C'_z \end{pmatrix}$$

the Jacobi matrix. Let $G = J^* J$. We have

$$||\varphi(\vec{x} + \Delta\vec{x}) - \varphi(\vec{x})||^2 = (J\Delta\vec{x}, J\Delta\vec{x}) + o(|\Delta\vec{x}|^2) = (\Delta\vec{x}, G\Delta\vec{x}) + o(|\Delta\vec{x}|^2).$$

Definition 1. A continuous differentiable one to one mapping

$$\varphi : \Omega_1 \to \Omega_2$$

of the domain $\Omega_1 \subset R^3$ on $\Omega_2 \subset R^3$ is conformal if for each point $\vec{x} \in \Omega_1$ there is a number $M(\vec{x})$ such that

$$||\varphi(\vec{x} + \Delta\vec{x}) - \varphi(\vec{x})|| = M(\vec{x})\,|\Delta\vec{x}| + o(|\Delta\vec{x}|).$$

Lemma 1. Let φ be continuous differentiable mapping with the Jacobi matrix J and $G = J^* J$. Then φ is conformal if and only if

$$27 \det(G) = \ \mathrm{tr}^3\,(G)\,.$$

Proof. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix G are nonnegative. The lemma's condition means that

$$\left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\right)^3 = \lambda_1 \lambda_2 \lambda_3$$

This equality is valid only if all eigenvalues are equal, i. e. $\lambda_1 = \lambda_2 = \lambda_3$.

Example. Let us consider the inverse transformation, which the point $(x,y,z) \neq (0,0,)$ maps to (A,B,C), where

$$(A,B,C) = \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2},\right)$$

We have

$$J = \begin{pmatrix} A'_x & A'_y & A'_z \\ B'_x & B'_y & B'_z \\ C'_x & C'_y & C'_z \end{pmatrix} =$$

$$= \frac{1}{(x^2+y^2+z^2)^2} \begin{pmatrix} -x^2+y^2+z^2 & -2xy & -2xz \\ -2xy & x^2-y^2+z^2 & -2yz \\ -2xz & -2yz & x^2+y^2-z^2 \end{pmatrix}$$

Consequently,

$$G = \frac{1}{(x^2+y^2+z^2)^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, lemma 1 condition satisfies and this mapping is conformal.

Definition 2. A quasi-conformal mapping is a continuous differentiable homeomorphism

$$\varphi : \Omega_1 \to \Omega_2$$

for which the ball of small radius maps to the ellipsoid the ratio of the main diagonals of which are uniformly bounded.

In this paper we introduce a new family of mappings, which are generalization of conformal mappings. For those mappings, which we named week - conformal, we have an analogy of Riemann's mapping theorem.

Definition 3. A week-conformal mapping is a continuous differentiable homomorphism

$$\varphi : \Omega_1 \to \Omega_2$$

for which the ball of small radius maps to the ellipsoid the main diagonals of which form geometric progression.

Lemma 2. Let φ be continuous differentiable mapping with Jacobi matrix J. Then it is week - conformal if and only if

$$\left(\operatorname{tr}^2(G) - |G|^2 \right)^3 = 8 \det(G) \operatorname{tr}^3(G),$$

where $G = J^*J$.

Proof. In terms of eigenvalues of the matrix $G = J^*J$, this condition one can write as follows

$$\lambda_1\lambda_2\lambda_3 \left(\lambda_1+\lambda_2+\lambda_3\right)^3 = \left(\lambda_1\lambda_2+\lambda_3\lambda_1+\lambda_3\lambda_2\right)^3.$$

So,

$$\left(\lambda_1\lambda_2\lambda_3 - \lambda_3^3\right)\left(\lambda_1+\lambda_2+\lambda_3\right)^3 - \left(\lambda_1\lambda_2+\lambda_3\lambda_1+\lambda_3\lambda_2\right)^3 + \left(\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_3^2\right)^3 = 0.$$

After simple transformations we get

$$(\lambda_1\lambda_2 - \lambda_3^2)\left((\lambda_1\lambda_2 + \lambda_3\lambda_1 + \lambda_3\lambda_2)\left(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2\right) - \lambda_1\lambda_2\left(\lambda_1 + \lambda_2 + \lambda_3\right)^2\right) = 0.$$

The last condition is equivalent to the following one

$$\left(\lambda_1\lambda_2 - \lambda_3^2\right)\left(\lambda_1\lambda_3 - \lambda_2^2\right)\left(\lambda_3\lambda_2 - \lambda_1^2\right) = 0.$$

Consequently, our condition means that eigenvalues of the matrix J^*J form a geometrical progression.

3 Green's function in R^2

In this section we define Green function and prove some of its properties.

Definition 4. Let Ω be a domain in R^2. A function $G(\vec{x}, \vec{y}), \quad \vec{x} \neq \vec{y} \in \Omega$ is called Green function for the domain Ω, if it satisfies the following conditions:

1. $G(\vec{x}, \vec{y})$ is continuous from below and

$$G(\vec{x}, \vec{y}) > 0, \quad \vec{x} \neq \vec{y} \in \Omega;$$

2. for each fixed point $\vec{y} \in \Omega$ there is a harmonic function $h(\vec{x}, \vec{y}), \quad \vec{x} \in \Omega$ such that

$$G(\vec{x}, \vec{y}) = \frac{1}{2\pi} \log \frac{1}{|\vec{x} - \vec{y}|} + h(\vec{x}, \vec{y});$$

3. if $u(\vec{x}), \quad \vec{x} \in \Omega$ is an arbitrary harmonic function satisfying the condition

$$u(\vec{x}) \leq G(\vec{x}, \vec{y}), \quad \vec{x} \in \Omega \setminus \{\vec{y}\},$$

then

$$u(\vec{x}) \leq 0, \quad \vec{x} \in \Omega.$$

In three dimension case, if $\Omega \subset R^3$, the Green's function one define by the same way replacing the second condition to the following one

$$G(\vec{x}, \vec{y}) = \frac{1}{4\pi|\vec{x} - \vec{y}|} + h(\vec{x}, \vec{y}), \quad \vec{x} \in \Omega \setminus \{\vec{y}\}.$$

It is well known, that if the boundary of a domain $\Omega \neq R^2$ has positif capacity, then it has unique Green function, see [4], p. 138. In particulary, if $\Omega \neq R^2$ is simply connected then it has Green function.

For a fixed point $\vec{y} \in \Omega$ and for an arbitrary number $0 < t < +\infty$ let us denote

$$\Omega_t = \{\vec{x}; \quad G(\vec{x}, \vec{y}) > t\}.$$

It is easy to note that for arbitrary value of $t > 0$ the domain Ω_t is connected and its Green function is $G(\vec{x}, \vec{y}) - t$.

Lemma 3. Let $u(\vec{x}), \quad \vec{x} \in \Omega$, be a harmonic function and

$$\{\vec{x}; \quad |\vec{x} - \vec{x}_0| \leq r\} \subset \Omega.$$

Let for some point $\vec{x}, \quad |\vec{x} - \vec{x}_0| = r$, we have

$$u(\vec{x}) = \inf\{u(\vec{y}); \quad |\vec{y} - \vec{x}_0| < r\}.$$

Then

$$|\nabla u(\vec{x})| \geq \frac{1}{2r}(u(\vec{x}_0) - u(\vec{x})).$$

Proof. For arbitrary $0 \leq \varphi < 2\pi$ and $0 < t < 1$ we have

$$\frac{r^2 - (rt)^2}{|re^{i\varphi} - t(\vec{x} - \vec{x}_0)|^2} \geq \frac{r^2 - (rt)^2}{(r + rt)^2} \geq \frac{1 - t}{2}.$$

So,

$$\frac{u(\vec{x}_0 + t(\vec{x} - \vec{x}_0)) - u(\vec{x})}{|\vec{x}_0 + t(\vec{x} - \vec{x}_0) - \vec{x}|} =$$

$$= \frac{1}{2\pi(r - rt)} \int_0^{2\pi} \frac{r^2 - (rt)^2}{|re^{i\varphi} - t(\vec{x} - \vec{x}_0)|^2}(u(\vec{x}_0 + re^{i\varphi}) - u(\vec{x}))d\varphi \geq$$

$$\geq \frac{1}{4\pi r} \int_0^{2\pi} (u(\vec{x}_0 + re^{i\varphi}) - u(\vec{x}))d\varphi = \frac{1}{2r}(u(\vec{x}_0) - u(\vec{x})).$$

Passing to the limit if $t \to 1 - 0$ we get the required result.

Remark. The analogous result is true in R^3.

Theorem 1. Let $G(\vec{x}, \vec{y})$ be Green function for a simply connected domain Ω in R^2. Then

$$|\nabla G(\vec{x}, \vec{y})| \neq 0, \quad \vec{x} \in \Omega \setminus \{\vec{y}\}.$$

Proof. Let us assume, that at some point $\vec{x}_0 \in \Omega$ we have

$$|\nabla G(\vec{x}_0, \vec{y})| = 0.$$

Denote

$$\Omega^+ = \{\vec{x}, \quad G(\vec{x}, \vec{y}) > G(\vec{x}_0, \vec{y})\}$$

and

$$\Omega^- = \{\vec{x}, \quad G(\vec{x}, \vec{y}) < G(\vec{x}_0, \vec{y})\}.$$

Let us note that the domain Ω^+ is connected. For sufficiently small number $\epsilon > 0$, such that

$$B(\vec{x}_0, \epsilon) \subset \Omega \setminus \{\vec{y}\}$$

holds, we consider the open set

$$A = B(\vec{x}_0, \epsilon) \setminus \{\vec{x}; \quad G(\vec{x}, \vec{y}) = G(\vec{x}_0, \vec{y})\}.$$

The set A can not consist of an odd number of components. Otherwise, we could find a point

$$\vec{x}_1 \in B(\vec{x}_0, \epsilon) \cap \{\vec{x}; \quad G(\vec{x}, \vec{y}) = G(\vec{x}_0, \vec{y})\}$$

in some neighborhood of which the function $G(\vec{x}, \vec{y}) - G(\vec{x}_0, \vec{y})$ would preserve its sign. So, $\vec{x}_1$ would be the point of local extremum, which is not possible for the nonconstant harmonic function $G(\vec{x}, \vec{y}), \quad \vec{x} \in B(\vec{x}_0, \epsilon)$. Moreover, the set A can not consist of two components. Indeed if it had two components then the boundary $\partial\Omega^+ \cap B(\vec{x}_0, \epsilon)$ would be smooth. Consequently, by lemma 1 we would have

$$|\nabla G(\vec{x}_0, \vec{y})| > 0.$$

Thus, the domain A consists at least with four connected components. This implies that the open set Ω^- consists of more than two connected components. Since our domain is simply connected, one of those components has the boundary, completely laying inside of $\partial\Omega^+$. On that connected component the function $G(\vec{x}, \vec{y})$ is identically constant end equal $G(\vec{x}_0, \vec{y})$. This is contradiction.

Let us note that in the theorem 1, the condition "simply connected", is essential. Indeed, for the domain $\{\vec{x}; \quad 1 < |\vec{x}| < 2\}$ the theorem 1 does not valid.

4 New prove of Riemann's theorem

In this section we give a new proof of Riemann's well known theorem on conformal mapping. In this prove we do not use methods of complex analysis. That is way it permits three dimensional analog.

Let us denote by

$$D(\vec{y}, r) = \{\vec{x}; \quad |\vec{x} - \vec{y}| < r\}.$$

Theorem 2. Let Ω be a simply connected domain in R^2. If $\Omega \neq R^2$ then there is a one to one conformal mapping

$$\varphi : \Omega \to D$$

of the domain Ω on the unit disk $D = D(\vec{0}, 1)$.

Proof. Let us fix a point $\vec{y} \in \Omega$ and $G(\vec{x}, \vec{y})$ be Green function of the domain Ω. Let us consider the following dynamical system in $\Omega \setminus \{\vec{y}\}$

$$\frac{d\vec{x}(t)}{dt} = -\frac{\nabla G(\vec{x}(t), \vec{y})}{2\pi |\nabla G(\vec{x}(t), \vec{y})|^2} e^{2\pi G(\vec{x}(t), \vec{y})}, \quad 0 < t < 1. \qquad (1)$$

For an arbitrary solution of this equation we have

$$\frac{d}{dt}\left(e^{-2\pi G(\vec{x}(t), \vec{y})}\right) = -2\pi e^{-2\pi G(\vec{x}(t), \vec{y})} \left(\nabla G(\vec{x}(t), \vec{y}), \frac{d\vec{x}(t)}{dt}\right) = 1.$$

Consequently,

$$G(\vec{x}(t), \vec{y}) = \frac{1}{2\pi} \ln \frac{1}{t}, \quad 0 < t < 1.$$

In the neighborhood of each point $\vec{x} \in \Omega \setminus \{\vec{y}\}$ the equation (1) has a unique solution passing through the point $\vec{x}$, see [1] p. 19.

In the neighborhood of the point $\vec{y}$ the equation (1) could be written in the following form

$$\frac{d\vec{x}(t)}{dt} = \frac{\vec{x}(t) - \vec{y}}{|\vec{x}(t) - \vec{y}|} \exp\{2\pi h(\vec{y}, \vec{y})\} + o(t), \quad t \to 0.$$

So, for each solution of our equation we have

$$\vec{x}(t) = \vec{y} + \vec{a} t \exp\{2\pi h(\vec{y}, \vec{y})\} + o(t), \quad t \to 0,$$

where $\vec{a}$ is a vector with norm one.

Consequently, for each point $x \in \Omega \setminus \{\vec{y}\}$ we can find a unique vector $\vec{a} = \vec{a}(\vec{x})$, such that there is a solution $\vec{x}(t)$ of our equation which passes through the point $\vec{x}$ and at the same time in the neighborhood of the point $\vec{y}$ satisfies the condition

$$\lim_{t \to 0} \frac{\vec{x}(t) - \vec{y}}{t} = \vec{a} \exp\{2\pi h(\vec{y}, \vec{y})\}.$$

Let us define the mapping

$$\varphi : \Omega \to D$$

as follows, $\varphi(\vec{y}) = 0$ and for the arbitrary point $\vec{x} \in \Omega \setminus \{\vec{y}\}$ we put

$$\varphi(\vec{x}) = \vec{a}(\vec{x}) e^{-2\pi G(\vec{x}, \vec{y})}.$$

It is obvious, that $\varphi(\vec{x})$ is a one to one mapping and $\varphi(\Omega) = D$.

Now we'll recall some facts about the constructed mapping, which permit to assert that it is conformal.

Let us take two solutions

$$\vec{x}(t), \quad \vec{x}_1(t)$$

of the equation (1). We denote by α the angle between the vectors $\vec{a}(\vec{x}(t))$ and $\vec{a}(\vec{x}_1(t))$. For arbitrary numbers $0 < t_0 < t_1 < 1$ let us denote by U the domain bounded by the curves

$$\gamma_1 = \{\vec{x}(t); \quad t_0 < t < t_1\}, \quad \gamma_2 = \{\vec{x}_1(t); \quad t_0 < t < t_1\}$$

and

$$\gamma_3 = \{\vec{x}; \quad G(\vec{x},\vec{y}) = G(x(\vec{t_0}),\vec{y})\}, \quad \gamma_4 = \{\vec{x}; \quad G(\vec{x},\vec{y}) = G(x(\vec{t_1}),\vec{y})\}.$$

Let $\vec{m}(\vec{x})$ be the unite outer normal to the boundary of the domain U at the point $\vec{x} \in \partial U$. For an arbitrary point $\vec{x} \in \gamma_1 \cup \gamma_2$ we have

$$\left(\frac{d\vec{x}(t)}{dt}, \vec{m}(\vec{x}(t)) \right) = 0, \quad t_0 < t < t_1.$$

Consequently,

$$(\nabla G(\vec{x},\vec{y}), \vec{m}(\vec{x})) = \frac{\partial G(\vec{x},\vec{y})}{\partial \vec{m}} = 0.$$

If $\vec{x} \in \gamma_3$ then we have $\vec{m}(\vec{x}) = -\vec{n}(\vec{x})$, where $\vec{n}(\vec{x})$ is the outer normal to the domain $\{\vec{x}; \quad G(\vec{x},\vec{y}) > t_0\}$. If $\vec{x} \in \gamma_4$ then we have $\vec{m}(\vec{x}) = \vec{n}(\vec{x})$, where $\vec{n}(\vec{x})$ is the outer normal to the domain $\{\vec{x}; \quad G(\vec{x},\vec{y}) > t\}$. Consequently,

$$\int_{\gamma_3} \frac{\partial G(\vec{x},\vec{y})}{\partial \vec{n}} ds = \int_{\gamma_4} \frac{\partial G(\vec{x},\vec{y})}{\partial \vec{n}} ds.$$

Passing to the limit we get

$$\alpha = 2\pi \lim_{t_0 \to +0} \int_{\gamma_3} \frac{\partial G(\vec{x},\vec{y})}{\partial \vec{n}} ds = 2\pi \int_{\gamma_4} \frac{\partial G(\vec{x},\vec{y})}{\partial \vec{n}} ds.$$

From definition of the mapping $\varphi(\vec{x})$ we have

$$|\varphi(\vec{x}(t)) - \varphi(\vec{x}_1(t)| = |t(\vec{x})||\vec{a}(\vec{x}(t)) - \vec{a}(\vec{x}_1(t))| =$$

$$= |t(\vec{x})| \left| 2\pi \int_{\gamma_4} \frac{\partial G(\vec{x},\vec{y})}{\partial \vec{n}} ds \right| =$$

$$= 2\pi \frac{\partial G(\vec{x}(t),\vec{y})}{\partial \vec{n}} \exp\{-2\pi G(\vec{x}(t),\vec{y})\} |\vec{x}(t) - \vec{x}_1(t)| + o(|\vec{x}(t) - \vec{x}_1(t)|).$$

Further on we can write

$$|\varphi(\vec{x}(t+\Delta t)) - \varphi(\vec{x}(t))| + o(|\Delta t|) = |\Delta t| + o(|\Delta t|) =$$

$$= |\vec{x}(t+\Delta t) - \vec{x}(t)| \left| \frac{d\vec{x}(t)}{dt} \right|^{-1} =$$

$$= 2\pi \frac{\partial G(\vec{x}(t), \vec{y})}{\partial \vec{n}} \exp\{-2\pi G(\vec{x}(t), \vec{y})\} |\vec{x}(t+\Delta t) - \vec{x}(t)| + o(|\Delta t|).$$

Let $r > 0$ be sufficiently small. We chose Δt and $\vec{x}_1(t)$ such that the equalities

$$|\vec{x}(t+\Delta t) - \vec{x}(t)| = |\vec{x}(t) - \vec{x}_1(t)| = r$$

holds. The vectors

$$\vec{x}(t+\Delta t) - \vec{x}(t)$$

and

$$\vec{x}(t) - \vec{x}_1(t)$$

are orthogonal. Consequently, the image of the disk $D(\vec{x}(t), r)$ is circle, in the first approach, once if the orthogonal vectors

$$\varphi(\vec{x}(t+\Delta t)) - \varphi(\vec{x}(t))$$

and

$$\varphi(\vec{x}(t)) - \varphi(\vec{x}_1(t))$$

satisfy the condition

$$|\varphi(\vec{x}(t+\Delta t)) - \varphi(\vec{x}(t))| = |\varphi(\vec{x}(t)) - \varphi(\vec{x}_1(t))| + o(r).$$

The last condition holds since

$$|\varphi(\vec{x}(t+\Delta t)) - \varphi(\vec{x}(t))| = 2\pi \frac{\partial G(\vec{x}(t), \vec{y})}{\partial \vec{n}} \exp\{-2\pi G(\vec{x}(t), \vec{y})\} r + o(r)$$

and

$$|\varphi(\vec{x}(t)) - \varphi(\vec{x}_1(t)| = 2\pi \frac{\partial G(\vec{x}(t), \vec{y})}{\partial \vec{n}} \exp\{-2\pi G(\vec{x}(t), \vec{y})\} r + o(r).$$

Remark. For constructed mapping at the points $\vec{x} \in \Omega \setminus \{\vec{y}\}$ we have

$$|\varphi'(\vec{x})| = 2\pi |\nabla G(\vec{x}, \vec{y})| \exp\{-2\pi G(\vec{x}, \vec{y})\}.$$

At the point $\vec{y}$ we have

$$|\varphi'(\vec{y})| = 2\pi \exp\{-2\pi h(\vec{y}, \vec{y})\}.$$

5 Green's function in $\mathbf{R}^3$

Definition 5. We say that a domain $\Omega \subset R^3$ is simply connected if

1. for an arbitrary bounded domain Ω_1 if we have $\partial\Omega_1 \subset \Omega$ then it follows $\Omega_1 \subset \Omega$;

2. an arbitrary closed curve laying in domain Ω permits continuous deformation in domain Ω to the point.

Lemma 4. Let Ω be a simply connected domain in R^3. Let Ω be a bounded and has smooth boundary, with Green function $G(\vec{x},\vec{y})$. Then

$$\nabla G(\vec{x},\vec{y}) \neq 0, \quad \vec{x} \in \Omega \setminus \{\vec{y}\}.$$

Proof. By lemma we have

$$\frac{\partial G(\vec{x},\vec{y})}{\partial n} \neq 0, \quad \vec{x} \in \partial\Omega.$$

Let $0 < t_0 < \infty$ be the biggest number for which there is a point $x_0 \in \Omega$ such that

$$\nabla G(\vec{x}_0,\vec{y}) = 0.$$

and $G(\vec{x}_0,\vec{y}) = t_0$. Denote

$$\Omega^+ = \{\vec{x}, \quad G(\vec{x},\vec{y}) > t_0\}$$

and

$$\Omega^- = \{\vec{x}, \quad G(\vec{x},\vec{y}) < t_0\}.$$

Let us note that the domain Ω^+ is connected. If Ω^- does not connected we come to the contradiction like of two dimensional case.

It turns out, that in three dimensional case, it is possible that the domain Ω^- is connected too.

In the domain

$$\{\vec{x}; \vec{x} \in \Omega,\, G(\vec{x},\vec{y}) > t_0\}$$

we consider the following dynamic system

$$\frac{d\vec{x}(t)}{dt} = -\frac{\nabla G(\vec{x}(t),\vec{y})}{4\pi|\nabla G(\vec{x}(t),\vec{y})|^2} G^2(\vec{x}(t),\vec{y}), \quad t_0 < t.$$

For an arbitrary solution of this equation we have

$$\frac{d}{dt}\frac{1}{G(\vec{x}(t),\vec{y})} = -\frac{1}{G^2(\vec{x}(t),\vec{y})}\left(\nabla G(\vec{x}(t),\vec{y}),\, \frac{d\vec{x}(t)}{dt}\right) = \frac{1}{4\pi}.$$

Consequently, we have

$$G(\vec{x}(t), \vec{y}) = \frac{1}{4\pi t}, \quad 0 < t < \infty.$$

So, for each $0 < \epsilon$, the solutions of this equation generate the following transformation

$$x(\infty) \to x(t_0 + \varepsilon)$$

which settle a one to one correspondence between the points of the manifolds $\partial\Omega$ and $\partial\Omega_{t_0+\epsilon}$. Consequently, those manifolds are homotopic equivalent.

In the domain

$$\{\vec{x}, \quad G(\vec{x}, \vec{y}) > t_0 + \varepsilon\}$$

there is a smooth closed curve γ_1, which passes throw the points x_0 and y.

For sufficiently small $\epsilon > 0$ the plain orthogonal to the curve γ_1 at the point x_0, cut a closed curve γ_2 on the boundary $\partial\Omega_{t_0+\epsilon}$ which have nonzero index in compare to the curve γ_1.

Since $\partial\Omega_{t_0+\epsilon}$ and $\partial\Omega$ are homotopic equivalent so, the curve γ_2, by continuous deformation, staying on the boundary $\partial\Omega_{t_0-\epsilon}$, is possible to transform to a point.

This is a contradiction since each curve on the boundary $\partial\Omega_{t_0+\epsilon}$ having sufficiently small diameter, has zero index in compare with the curve γ_1.

6 Week conformal mapping in $\mathbf{R}^3$

In this paper we prove the main result of our paper.

Theorem 3. Let Ω be a simply connected domain in R^3. If Ω is a bounded and has smooth boundary then there is a one to one week - conformal mapping

$$\varphi : \Omega \to B$$

of the domain Ω onto the unit ball $B = \{x \in R^3; \quad |x| < 1\}$.

Proof. We consider the following dynamic system

$$\frac{d\vec{x}(t)}{dt} = -\frac{\nabla G(\vec{x}(t), \vec{y})}{4\pi|\nabla G(\vec{x}(t), \vec{y})|^2} G^2(\vec{x}(t), \vec{y}), \quad 0 < t < \infty.$$

In neighborhood of the point $\vec{y}$ we have

$$-\frac{\nabla G(\vec{x}, \vec{y})}{4\pi|\nabla G(\vec{x}, \vec{y})|^2} G^2(\vec{x}, \vec{y}) =$$

$$= \left(\frac{\vec{x}-\vec{y}}{4\pi|\vec{x}-\vec{y}|^3} - \nabla h\right)\left(\frac{1}{4\pi|\vec{x}-\vec{y}|} + h\right)^2$$

$$\left(\frac{1}{16\pi^2|\vec{x}-\vec{y}|^4} - \frac{(\vec{x}-\vec{y},\, \nabla h)}{2\pi|\vec{x}-\vec{y}|^3} + |\nabla h|^2\right)^{-1} =$$

$$= \left(\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|} - 4\pi|\vec{x}-\vec{y}|^2\nabla h\right) \frac{(1+4\pi|\vec{x}-\vec{y}|h)^2}{1 - 8\pi|\vec{x}-\vec{y}|(\vec{x}-\vec{y},\, \nabla h) + 16\pi^2|\vec{x}-\vec{y}|^4|\nabla h|^2} =$$

$$= \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|} + 8\pi h(\vec{y},\vec{y})(\vec{x}-\vec{y}) + O(|\vec{x}-\vec{y}|^2)$$

So, for each solution of our equation we have

$$\vec{x}(t) = \vec{y} + \vec{a}t + 4\pi\vec{a}t^2 h(\vec{y},\vec{y}) + o(t^2), \quad t \to 0,$$

where $\vec{a}$ is a vector with norm one.

Consequently, for each point $\vec{x} \in \Omega\setminus\{\vec{y}\}$ we can find the unique vector $\vec{a} = \vec{a}(\vec{x})$ of unit norm, such that a solution $\vec{x}(t)$ passes through the point $\vec{x}$ and

$$\lim_{t\to 0} \frac{\vec{x}(t) - \vec{y}}{t} = \vec{a}.$$

By definition the vector $\vec{a}(\vec{x}(t))$ is the same for all values of $0 < t < \infty$.

Let $\vec{x} = \vec{x}(t_0)$. We denote by

$$\gamma(\vec{x}) = \{\vec{x}(t); \quad t_0 \le t < \infty\}$$

the curve begins of the point $\vec{x}$ and goes to the boundary of the domain Ω.

If for each point $\vec{x} \in \Omega$ the curve $\gamma(\vec{x})$ has a finite length, then we can define the mapping

$$\varphi : \Omega \to B$$

as follows, $\varphi(\vec{y}) = 0$ and for the arbitrary point $\vec{x} \in \Omega \setminus \{\vec{y}\}$ we put

$$\varphi(\vec{x}) = \vec{a}(\vec{x}) \exp\left\{-\int_{\gamma(\vec{x})} \sqrt{4\pi|\nabla G(\vec{z},\vec{y})|}ds(\vec{z})\right\}.$$

It is obvious, that φ is a one to one mapping onto the unit ball B.

Now let us consider the properties of the constructed mapping.

For arbitrary nonzero vector $\vec{a}$ let us denote by $D_\alpha(\vec{a})$ the round cone of bisector $\vec{a}$ and the sector

$$\{\vec{y}; \quad |\vec{y}| = 1,\ \vec{y} \in D_\alpha(\vec{a})\}$$

has area equal α.

Let us fix a point $\vec{x} \in \Omega$ and a number $0 < \alpha < 4\pi$. For arbitrary numbers $0 < t_0 < t_1 < \infty$ let us denote by U the domain

$$U = \bigcup_{\vec{a}(\vec{x}(t_0)) \in D_\alpha(\vec{a}(\vec{x}))} \{\vec{x}(t); \quad t_0 < t < t_1\}$$

We denote

$$W(t_0, \alpha) = \{\vec{x}(t_0); \quad \vec{a}(\vec{x}(t_0)) \in D_\alpha(\vec{a}(\vec{x}))\}$$

and

$$W(t_1, \alpha) = \{\vec{x}(t_1); \quad \vec{a}(\vec{x}(t_1)) \in D_\alpha(\vec{a}(\vec{x}))\}$$

By Green's formula we have

$$\int_{W(t_0,\alpha)} \frac{\partial G(\vec{z}, \vec{y})}{\partial \vec{n}} ds(\vec{z}) = \int_{W(t_1,\alpha)} \frac{\partial G(\vec{z}, \vec{y})}{\partial \vec{n}} ds(\vec{z}).$$

Passing to the limit if $t_0 \to 0$ we get

$$\int_{W(t_1,\alpha)} \frac{\partial G(\vec{z}, \vec{y})}{\partial \vec{n}} ds(\vec{z}) = \lim_{t_0 \to +0} \int_{W(t_0,\alpha)} \frac{\partial G(\vec{z}, \vec{y})}{\partial \vec{n}} ds(\vec{z}) = \frac{\alpha}{4\pi}.$$

Consequently, for small α we have

$$s(W(t_1, \alpha)) \frac{\partial G(\vec{x}(t_1), \vec{y})}{\partial \vec{n}} = \frac{\alpha}{4\pi} + o(\alpha).$$

The vector $\vec{x}(t_1 + \Delta t) - \vec{x}(t_1)$ is orthogonal to the surface $W(t_1, \alpha)$ and

$$|\vec{x}(t_1 + \Delta t) - \vec{x}(t_1)| = \left| \frac{dx(t_1)}{dt} \right| |\Delta t| = \frac{G^2(x(t_1), y)}{4\pi |\nabla G(x(t_1), y)|} |\Delta t| + o(|\Delta t|).$$

We choose the parameters α and Δt such that

$$s(W(t_1, \alpha)) = \pi |\vec{x}(t_1 + \Delta t) - \vec{x}(t_1)|^2.$$

This means that we have

$$\frac{G^4(x(t_1), y)}{16\pi |\nabla G(x(t_1), y)|^2} |\Delta t|^2 = \frac{\alpha}{4\pi |\nabla G(x(t_1), y)|} + o(\alpha).$$

The vector $\varphi(\vec{x}(t_1 + \Delta t) - \varphi(\vec{x}(t_1))$ is orthogonal to the surface $\varphi(W(t_1, \alpha))$.

Let us note that the image of the subset $W(t_1, \alpha)$ is a round sector in the sphere of the center at the point $\vec{0}$ and of the radius equals $|\varphi(\vec{x}(t_1))|$. So, we have

$$s(\varphi(W(t_1, \alpha))) = \alpha |\varphi(\vec{x}(t_1))|^2.$$

Since

$$|\varphi(\vec{x}(t_1))| = \exp\left\{-\int_{\gamma(\vec{x}(t_1))} \sqrt{4\pi|\nabla G(\vec{z},\vec{y})|}ds(\vec{z})\right\} =$$

$$= \exp\left\{-\int_{t_1}^{\infty} \frac{G^2(x(t),y)}{\sqrt{4\pi|\nabla G(x(t),y)|}}dt\right\}$$

So,

$$|\varphi(\vec{x}(t_1+\Delta t)) - \varphi(\vec{x}(t_1))| = |\varphi(\vec{x}(t_1))|\frac{G^2(\vec{x}(t_1),\vec{y})}{\sqrt{4\pi|\nabla G(\vec{x}(t_1),\vec{y})|}}|\Delta t| + o(|\Delta t|).$$

Consequently, we have

$$s(\varphi(W(t_1,\alpha))) = \alpha|\varphi(\vec{x}(t))|^2 = \pi|\varphi(\vec{x}(t+\Delta t)) - \varphi(\vec{x}(t))|^2 + o(|\Delta t|^2).$$

This relation is equivalent to the week - conformal condition at the point $\vec{x} \in \Omega$ for constructed mapping.

Remark. For an arbitrary point $\vec{x} \neq \vec{y}$ we have

$$\lim_{\Delta t\to 0} \frac{|\varphi(\vec{x}(t+\Delta t)) - \varphi(\vec{x}(t))|}{|\vec{x}(t+\Delta t) - \vec{x}(t)|} =$$

$$= \sqrt{4\pi|\nabla G(\vec{x},\vec{y})|}\exp\left\{-\int_{\gamma(\vec{x})} \sqrt{4\pi|\nabla G(\vec{z},\vec{y})|}ds(\vec{z})\right\}$$

where $\vec{x} = \vec{x}(t)$. So, we have

$$\varphi(\vec{x}) = \varphi(\vec{x}) - \varphi(\vec{y}) = \vec{x} - \vec{y} + o(|\vec{x}) - \vec{y}|).$$

References

1. Ph. Hartman, Ordinary differential equations, New York, London, Sydney 1964.

2. B. A. Dubrovin, S. P. Novicov, A. T. Fomenco, Modern geometry, Moscow 1986.

3. V. Arnold, A. Varchenko, S. Gusein - Zade, The singularities of differentiable mappings, Moscow 1982

4. W. Hayman, P. Kennedy, Subharmonic functions, London, New York, San Francisco, 1976.

5. T. Ransford, Potential theory in the complex plane, Cambridge University press 1995.

6. L. Ahlfors, Mobius transformations in several dimensions, University of Minnesota 1981.