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On non-linear discrete boundary-value problems and semi-groups

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Abstract

The following discrete boundary - value problem for non-linear system

$$x_k = \varphi_k(x_{k-1}, y_k),\ y_{k-1} = \psi_k(x_{k-1}, y_k),\ k = \overline{1, N},\ \ N < \infty,\ \ x_0 = a,\ \ y_N = b,$$

is considered. Here the functions $\varphi_k(x, y)$, $\psi_k(x, y) \geq 0$ are monotone with respect to arguments $x, y \geq 0$, satisfing the condition of dissipativity or conservativity: $\varphi_k(x, y) + \psi_k(x, y) \leq x + y + \gamma_k$, as well as two simple additional conditions. A relation of this problem with multistep processes is demonstrated. Existence and uniqueness of minimal solution of problem is proved. A semi-group approach to solving of problem is developed. The approach is adjoined with V. Ambartsumian Principle of Invariance and R.Bellman method of Invariant Imbedding.

Key words: Discrete boundary-value problem, solvability, Invariant Imbedding.

0. Introduction

Let Ω_0 be the set of ordered pairs (φ, ψ) of non-negative functions $\varphi(x, y)$ and $\psi(x, y)$ on $R^2_+ = [0, \infty) \times [0, \infty)$. The element $(\varphi, \psi) \in \Omega_0$ may be considered as a Black Box (BB) $\Pi = \Pi(\varphi, \psi)$ having two entries and two exits (conditionally left and right). If x and y are the entire impulses from the left and the right, then $\psi(x, y)$ and $\varphi(x, y)$ are the exit impulses from left and right respectively (see Fig.1).

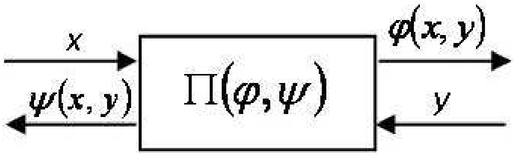

Fig.1

Consider the following system of non-linear equations with respect to $(x_k)_{k=1}^N$, $(y_k)_{k=0}^{N-1}$, $x_k, y_k \geq 0$, $N < \infty$:

$$\begin{aligned} x_k &= \varphi_k(x_{k-1}, y_k) \\ y_{k-1} &= \psi_k(x_{k-1}, y_k) \\ k &= 1, 2, \cdots, N. \end{aligned} \tag{0.1}$$

with boundary conditions:

$$x_0 = x, \tag{0.2}$$

$$y_N = y, \tag{0.3}$$

where $x, y \geq 0, \quad (\varphi_k, \psi_k) \in \Omega_0$.
By its structure, the system (0.1) corresponds to multistep processes with mutual connection (see for example [1]). The pair φ_k, ψ_k determines the k-th step $\Pi_k = \Pi_k(\varphi_k, \psi_k)$ of process. It is easy to see from Fig. 2, that considering multistep process is described by boundary-value problem (0.1)-(0.3).

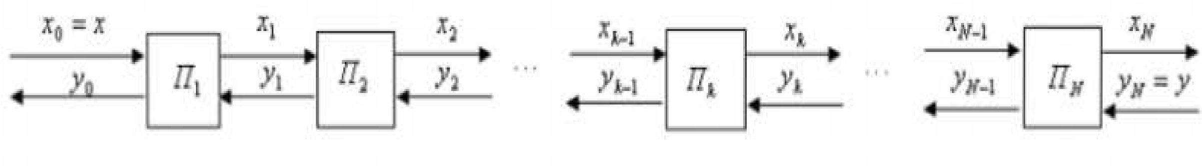

Fig.2

The problem (0.1)-(0.3) may be served as a mathematical model for non-linear radiative transfer processes.
The problem (0.1)-(0.3) will be supplemented with the condition of minimality of solution. Existence and uniqueness of minimal solution of problem (0.1)-(0.3) will be proved. A semi-group approach to the solution of problem is developed. This approach is adjoined with V.Ambartsumian Principle of Invariance and R.Bellman method of Invariant Imbedding (see [2]-[7]).
The following functions $\varphi(x, y)$ and $\psi(x, y)$, defined by

$$\varphi : (x, y) \to x_N, \qquad \psi : (x, y) \to y_0 \tag{0.4}$$

are of special interest. These functions determine an element $\Pi(\varphi, \psi) \in \Omega_0$.
We shall show, that the set (1.1)-(1.4) of basic properties of functions φ_k, ψ_k not only ensure solvability of problem (0.1)-(0.3), but also go over to the functions (0.4). The Semi-group structure in the set $\{\Pi_k\} \subset \Omega_0$ of BB, possessing properties (1.1)-(1.4) , will be introduced.

1. Statement of problem.

1.1. The Main problem and Exit problem. Under minimal solution (MS) of boundary-value problem (0.1)-(0.3) we shall assume a (positive) solution $\{\breve{x}_k, \breve{y}_k\}$, satisfying the condition $\breve{x}_k \leq x_k$, $\breve{y}_k \leq y_k$, where $\{x_k, y_k\}$-is an arbitrary solution of this problem. If MS exists, then it is unique.
The Main problem of this paper is to construct the Minimal solution of problem (0.1)-(0.3).
The Exit problem (the reflection-transmission problem) is to determine y_0 and x_N by given x_0, y_N in condition of the solvability of the Main problem.

The Exit problem is of well known interest in Radiative Transfer and etc. In this paper it will be shown, that, analogous to linear case, the Exit problem can be solved separately, without solution of the Main problem. The solution of the Exit problem will be used for the solution of the Main problem, using one recurrent procedure.

1.2. On condition of minimality of solution. The minimal solution of equations of the form $f = Af$ with positive operator A is of interest in the theory of non-linear operator equations, in the theory of Stochastic processes, in Radiative Transfer and etc. (see [7]-[9]). In some problems the condition of minimality corresponds to the equilibrium (stationary) distributions, which are the limit of time-dependent processes when $t \to \infty$.

1.3. The classes $\widetilde{\Omega}$ and Ω of black boxes. Denote by $\widetilde{\Omega} \subset \Omega_0$ the set of elements $\Pi = \Pi(\varphi, \psi) \in \Omega_0$, satisfying the following two conditions (1.1)-(1.2):
i) Monotonicity: the functions φ, ψ increase on $[0, \infty) \times [0, \infty)$ by each of arguments:

$$\varphi(x, y), \;\; \psi(x, y) \uparrow \; \text{ by } x, \, y. \tag{1.1}$$

ii) Continuity from below:

$$\text{if } \; x^{(n)} \uparrow x, \;\; y^{(n)} \uparrow y, \;\text{ then } \varphi\left(x^{(n)}, y^{(n)}\right) \to \varphi(x, y), \;\; \psi\left(x^{(n)}, y^{(n)}\right) \to \psi(x, y). \tag{1.2}$$

Denote by $\Omega \subset \widetilde{\Omega}$ the class of elements $\Pi = \Pi(\varphi, \psi) \in \widetilde{\Omega}$, satisfying the following two additional conditions (1.3)-(1.4).

$$\text{iii) If } \; y \to \infty, \;\text{ then } \; \psi(x, y) \to \infty. \tag{1.3}$$

iv) Condition of conservativity or dissipativity:

$$\varphi(x, y) + \psi(x, y) \leq x + y + \gamma, \tag{1.4}$$

where $\gamma \geq 0$ is constant, which we shall call the power of internal sources of BB.

The element $\Pi(\varphi, \psi)$ we shall call conservative, with source of power γ, if the following equality takes place in (1.4):

$$\varphi(x, y) + \psi(x, y) = x + y + \gamma, \tag{1.5}$$

The element $\Pi(\varphi, \psi)$ we shall call uniformly dissipative, if the following more strong inequality instead of (1.4) takes place:

$$\varphi(x, y) + \psi(x, y) \leq q(x + y) + \gamma, \quad 0 < q < 1. \tag{1.6}$$

Denote, that the condition (1.3) may be changed by condition

$$\varphi(x, y) \to \infty \;\; if \;\; x \to \infty. \tag{1.7}$$

2. Some general properties of the solution of the problem (0.1)-(0.3).

2.1. The mail solution of the problem (0.1)-(0.3). Consider the problem (0.1)-(0.3) under assumption, that each of pairs φ_k, ψ_k satisfies the conditions (1.1)-(1.2), i.e. $\{\Pi_k(\varphi_k, \psi_k)\}_{k=1}^N \subset \widetilde{\Omega}$. Consider the following simple iterations for (0.1)-(0.3):

$$\begin{aligned} x_k^{(n+1)} &= \varphi_k\left(x_{k-1}^{(n)}, y_k^{(n)}\right), \\ y_{k-1}^{(n+1)} &= \psi_k\left(x_{k-1}^{(n)}, y_k^{(n)}\right), \; n = 0, 1, 2, \cdots, \\ x_0^{(n)} &= x, \\ y_N^{(n)} &= y. \end{aligned} \tag{2.1}$$

$$x_k^{(0)} = 0,\; k = 1,\, 2,\, \cdots,\, N,\; y_k^{(0)} = 0,\; k = 0, 1,\, 2,\, \cdots,\, (N-1). \tag{2.2}$$

Applying the induction by n one can show, that the iteration sequences $\left\{x_k^{(n)}, y_k^{(n)}\right\}$ increase (component by component) with n:

$$x_k^{(n)} \uparrow,\; y_k^{(n)} \uparrow,\; \text{with } n. \tag{2.3}$$

Let the sequences $x_k^{(n)}$ and $y_k^{(n)}$ converge (k is fixed):

$$x_k^{(n)} \to \widetilde{x_k},\; y_k^{(n)} \to \widetilde{y_k}.$$

In accordance with (1.2), one can take the limit in each of equalities (2.1), hence $\{\widetilde{x_k}, \widetilde{y_k}\}$ satisfies the problem (0.1)-(0.3). We shall call this solution the Basic solution (BS) of the problem. The following lemma takes place.

Lemma 2.1. *Let $\{\Pi_k\}_{k=1}^N \subset \widetilde{\Omega}$ and the sequences $x_k^{(n)}$ and $y_k^{(n)}$, determined by* (2.1),(2.2) *are bounded. Then there exist the BS of the problem* (0.1)-(0.3).

Lemma 2.2.
i) Let $\{\Pi_k\}_{k=1}^N \in \widetilde{\Omega}$. Assume that the solution of the problem (0.1)-(0.3) *exists for any $x = a,\; y = b$, where $a, b \geq 0$. Then the Basic solution of this problem exists for arbitrary boundary conditions $x \leq a,\; y \leq b$.*
ii) The BS is the minimal solution of the problem.
iii) The BS depends on boundary values x, y monotonically (component by component).

Proof. Let $\{x_k, y_k\}$ be some (positive) solution of the problem (0.1)-(0.3) when $x = a,\; y = b$. Consider the iterations (2.1) for some $x, y,\; x \leq a,\, y \leq b$. Using the induction by n one can check that: $x_k^{(n)} \leq x_k, y_k^{(n)} \leq y_k$. From here and from (2.3) follows the monotonic convergence of these sequences:

$$x_k^{(n)} \uparrow \widetilde{x_k} \leq x_k,\; y_k^{(n)} \uparrow \widetilde{y_k} \leq y_k. \tag{2.4}$$

In accordance with Lemma 2.1, $(\widetilde{x_k}, \widetilde{y_k})$ is a solution of the problem. It follows from (2.4), that $(\widetilde{x_k}, \widetilde{y_k})$ is the Minimal solution of the problem (2.1),(2.2) when $x = a,\; y = b$. As an

a, b one can take arbitrary boundary conditions so, that the problem (0.1)-(0.3) is solvable. Hence for arbitrary $x \le a$, $y \le b$ the corresponding set $(\widetilde{x_k}, \widetilde{y_k})$ represents the Minimal solution of the problem. It also follows from (2.4), that $(\widetilde{x_k}, \widetilde{y_k})$ monotonically depends on boundary values x, y. The lemma is proved.

2.2. Addition in $\widetilde{\Omega}$. Let $\{\Pi_k\} \subset \widetilde{\Omega}$. Assume that the Minimal solution of the problem (0.1)-(0.3) exists for arbitrary $x, y \ge 0$. Then the ordered set $\{\Pi_k\}$ determines the exit functions φ, ψ in accordance with (0.4). The element $\Pi(\varphi, \psi) \in \Omega_0$ will be called the minimal sum (or sum) of elements $\Pi_1, \cdots, \Pi_N$:

$$\Pi = \oplus \{\Pi_1, \cdots, \Pi_N\}. \tag{2.5}$$

Let us show, that $\Pi(\varphi, \psi) \in \widetilde{\Omega}$.

Lemma 2.3. *Let* $\{\Pi_k\}_{k=1}^N \in \widetilde{\Omega}$ *and the problem* (0.1)-(0.3) *has the minimal solution* $\forall\, x \ge 0$, $y \ge 0$. *Then the functions* φ, ψ *defined by* (0.4) *possess the properties* (1.1) *and* (1.2), *i.e.* $\Pi(\varphi, \psi) = \oplus \{\Pi_1, \cdots, \Pi_N\} \in \widetilde{\Omega}$.

Proof. Monotonic dependence of $\varphi(x, y)$ and $\psi(x, y)$ on x and y follows from assertion iii) of Lemma 2.2. Let $\{x_k, y_k\}$ be the minimal solution of the problem (0.1)-(0.3), $x^{(n)} \uparrow x$, $y^{(n)} \uparrow y$. Denote by $x_k^{(n)}, y_k^{(n)}$ the minimal solution of the corresponding problem, with $x = x^{(n)}$, $y = y^{(n)}$. We have $x_k^{(n)} \uparrow \widetilde{x_k} \le x_k$, $y_k^{(n)} \uparrow \widetilde{y_k} \le y_k$. Taking the limit in equations for $x_k^{(n)}, y_k^{(n)}$, we make sure that $\{\widetilde{x_k}, \widetilde{y_k}\}$ satisfies the problem (0.1)-(0.3). It follows from (0.4) and from minimality of $\{x_k,\, y_k\}$, that $\{\widetilde{x_k}, \widetilde{y_k}\} = \{x_k,\, y_k\}$. The Lemma is proved.

3. The existence of the solution of the Main problem.

3.1. The case of fulfilment of the condition (1.3)**.** Consider the procedure of addition in $\widetilde{\Omega}$ under the assumption that the summands Π_k in (2.5) satisfied the condition (1.3). Then the Lemma 2.1 may be strengthen.

We shall use the following simple fact: if the function ψ satisfies the conditions (1.1), (1.3), the sequence $x^{(n)}$ increases and $y^{(n)} \uparrow \infty$, then

$$\psi\left(x^{(n)}, y^{(n)}\right) \uparrow \infty. \tag{3.1}$$

Lemma 3.1. *Let the elements* $\Pi_k \in \widetilde{\Omega}$, $k = 1, \cdots, N$ *satisfy the condition* (1.3)*. Let the sequence* $y_0^{(n)}$, *defined by* (2.1),(2.2) *for some* $x = a$, $y = b$, $a, b \ge 0$, *is bounded. Then for* $x \le a$, $y \le b$ *all of the sequences* $x_k^{(n)}, y_k^{(n)}$ *are bounded and the iterations* (2.1),(2.2) *converge to the minimal solution of the problem* (0.1)-(0.3).

Proof. In accordance with Lemma 2.2, it is sufficient to consider only the case where $x = a$, $y = b$. The rule of contraries will be used. Let $k \ge 1$ be the minimal number such, that only if one of sequences $x_k^{(n)}, y_k^{(n)}$ are non-bounded. Let $y_k^{(n)} \uparrow \infty$. Then from the equalities $y_{k-1}^{(n+1)} = \psi_k\left(x_{k-1}^{(n)}, y_k^{(n)}\right)$ (see (2.1) and (3.1) we shall have $y_{k-1}^{(n)} \uparrow \infty$, which is in contradiction with the definition of number k. Consider now the sequence $x_k^{(n)}$. In accordance with (2.1) we have $x_k^{(n+1)} = \varphi_k\left(x_{k-1}^{(n)}, y_k^{(n)}\right)$. From here and from boundedness of $x_{k-1}^{(n)}, y_k^{(n)}$ follows boundedness of $x_k^{(n+1)}$. The lemma is proved.

3.2. Existence of the solution. Consider the problem (0.1)-(0.3) assuming that the elements $\Pi_k(\varphi_k, \psi_k) \in \Omega$, i.e. they satisfy the conditions (1.1)-(1.4).

In accordance with (1.4) we have:

$$\varphi_k(x, y) + \psi_k(x, y) \leq x + y + \gamma_k, \quad \gamma_k \geq 0. \tag{3.2}$$

Lemma 3.2. *Let $\Pi_k(\varphi_k, \psi_k) \in \Omega$, $k = 1, \cdots, N$, the and the iterative sequences $x_k^{(n)}, y_k^{(n)}$ are defined by* (2.1),(2.2). *Then the following estimate holds:*

$$y_0^{(n)} + x_N^{(n)} \leq x + y + (\gamma_1 + \gamma_2 + \cdots \gamma_N). \tag{3.3}$$

where the constants $\gamma_1 \cdots, \gamma_N$ are determined by (3.2).

Proof. Adding all of the relations (2.1) and using (3.2) we get:

$$\sum_{k=1}^{N}\left[x_k^{(n+1)} + y_{k-1}^{(n+1)}\right] = \sum_{k=1}^{N}\left[\varphi_k\left(x_{k-1}^{(n1)}, y_k^{(n)}\right) + \psi_k\left(x_{k-1}^{(n1)}, y_k^{(n)}\right)\right] \leq \sum_{k=1}^{N}\left[x_{k-1}^{(n)} + y_k^{(n)} + \gamma_k\right].$$

From here, taking into account (2.2), we obtain

$$\sum_{k=1}^{N-1}\left[x_k^{(n+1)} + y_k^{(n+1)}\right] + x_N^{(n+1)} + y_0^{(n+1)} \leq \sum_{k=1}^{N-1}\left[x_k^{(n)} + y_k^{(n)}\right] + x + y + \sum_{k=1}^{N}\gamma_k.$$

Using monotonicity of sequences $x_k^{(n)}, y_k^{(n)}$ by n we come to (3.3). The lemma is proved.

The boundedness of sequence $y_0^{(n)}$ follows from the estimate (3.3). In accordance with Lemma 3.1, the problem (0.1)-(0.3) possesses the Minimal solution. Using the Lemma 2.3 we come to the existence of minimal sum $\Pi(\varphi, \psi) = \oplus\Pi_1, \cdots, \Pi_N \in \widetilde{\Omega}$, defined by (2.5). Let us show, that $\Pi \in \Omega$. The fulfilment of property (1.4), with $\gamma = \sum\limits_{k=1}^{N}\gamma_k$ follows from the estimate (3.3). Let us check the property (1.3). Let $y_N^{n)} \uparrow \infty$ at $n \to \infty$. Let now $\left\{x_k^{(n)}, y_k^{(n)}\right\}$ be the minimal solution of the problem (0.1)-(0.3), in which $y = y_N^{(n)}$. Let us show, that $y_k^{(n)} \to \infty$, $k = 0, 1, \cdots, (N-1)$. We shall use the rule of contraries. Denote by m the largest value of k, for which the sequence $y_k(n)$ remains bounded. Because of $y_N^{(n)} \uparrow \infty$ we have $m < N$. We have $y_m^{(n)} = \psi_{m+1}\left(x_m^{(n)}, y_{m+1}^{(n)}\right)$. It follows from the property (1.3) for ψ_{m+1} and (3.1), that $y_m^{(n)} \uparrow \infty$, which is in contradiction with the definition of m.

The following theorem of existence and uniqueness have been proved.

Theoram 3.1. *Let* $\Pi_k(\varphi_k, \psi_k) \in \Omega,\ k = 1, \cdots, N$. *Then the boundary-value problem* (0.1)-(0.3) *possesses unique Minimal solution, which is the limit of simple iterations, defined by* (2.1),(2.2). *The following inequality takes place:*

$$x_N + y_0 \le x + y + (\gamma_1 + \gamma_2 + \cdots + \gamma_N). \tag{3.4}$$

The sum $\Pi(\varphi, \psi) = \oplus \Pi_1, \cdots, \Pi_N \in \widetilde{\Omega}$ *possesses properties* (1.1)-(1.4), *i.e.* $\Pi \in \Omega$.

4. The semi-group approach to the solution of the Exit problem and the Main problem.

4.1. The semi-group Ω. The present section is devoted to the dissemination of V.Ambartsumian method of addition of layers (see [2],[3],[6],[7]) to non-linear boundary-value problems. Our approach has certain likeness with R. Bellman method of Invariant Imbedding.

It follows from the theorem 3.1, that by equality

$$\Pi_1 \oplus \Pi_2 = \oplus \{\Pi_1, \Pi_2\}$$

a binary operation (of addition) $\oplus$ in Ω is determined.

It is easy to check in simple examples, that the operation $\oplus$ is non-commutative.

Consider the functions

$$\varphi_0(x, y) = x,\ \psi_0(x, y) = y.$$

These functions determine the element $\theta = \Pi_0(\varphi_0, \psi_0) \in \Omega$ so, that

$$\Pi \oplus \theta = \theta \oplus \Pi = \Pi,\ \ \forall \Pi \in \Omega. \tag{4.1}$$

Let us show, that the class Ω represents a semi-group relatively the operation $\oplus$. For this we must show, that this operation is associative.

Let $\Pi_i \in \Omega,\ i = 1, 2, 3$. In accordance with the theorem 3.1 there exist the elements $\Pi_1 \oplus (\Pi_2 \oplus \Pi_3)$, $(\Pi_1 \oplus \Pi_2) \oplus \Pi_3$, $\oplus \Pi_1, \Pi_2, \Pi_3 \in \Omega$. Let us show, that

$$\Pi_1 \oplus (\Pi_2 \oplus \Pi_3) = (\Pi_1 \oplus \Pi_2) \oplus \Pi_3 = \oplus \Pi_1, \Pi_2, \Pi_3. \tag{4.2}$$

The sum $\Pi_1 \oplus (\Pi_2 \oplus \Pi_3)$ give a solution of the problem (0.1)-(0.3) (where $N = 3$) by the following way. We have $\Pi_1 \oplus (\Pi_2 \oplus \Pi_3) = \Pi_1 \oplus \widetilde{\Pi}$, where $\widetilde{\Pi}(\widetilde{\varphi}, \widetilde{\psi}) = \Pi_2 \oplus \Pi_3 \in \Omega$. Consider the system

$$\widetilde{x_1} = \varphi_1(x, \widetilde{y_1}), \tag{4.3}$$

$$\widetilde{y_1} = \widetilde{\psi}(\widetilde{x_1}, y), \tag{4.4}$$

where x and y are the numbers, participating in the boundary conditions (0.2),(0.3). Because of existence of the sum $\Pi_1 \oplus \widetilde{\Pi}$, this system has the minimal solution $\widetilde{x_1}, \widetilde{y_1}$. Let $\widetilde{x_2}, \widetilde{y_2}$ be the minimal solution of the system

$$\widetilde{x_2} = \varphi_2(\widetilde{x_1}, \widetilde{y_2}),\ \ \widetilde{y_2} = \psi_3(\widetilde{x_2}, y). \tag{4.5}$$

The following equality takes place:

$$\widetilde{y_1} = \psi_2(\widetilde{x_1}, \widetilde{y_2}). \tag{4.6}$$

Introduce $\widetilde{y_0}$, $\widetilde{x_3}$ by following equalities:

$$\widetilde{x_3} = \varphi_3(\widetilde{x_2}, y), \;\; \widetilde{y_0} = \psi_1(x, \widetilde{y_1}). \tag{4.7}$$

The relations (4.2),(4.4)-(4.6) coincide with (0.1)-(0.3). Hence $\widetilde{x_k}, \widetilde{y_k}$ is a solution of the problem (0.1)-(0.3). Let us show, that the constructed solution $\widetilde{x_k}, \widetilde{y_k}$ coincides with the minimal solution x_k, y_k of the problem. Consider the following iterations for (4.3):

$$\widetilde{x_1}^{(n+1)} = \varphi_1\left(x, \widetilde{y_1}^{(n)}\right), \;\; \widetilde{y_1}^{(n+1)} = \widetilde{\psi}\left(\widetilde{x_1}^{(n)}, y\right), \;\; \widetilde{x_1}^{(0)} = \widetilde{y_1}^{(0)} = 0, \;\; n = 0, 1, \cdots$$

Denote by $\widetilde{x_2}^{(n)}, \widetilde{y_2}^{(n)}$ the minimal solution of the system (4.4), in which $\widetilde{x_1}$ is changed by $\widetilde{x_1}^{(n)}$ and $\widetilde{y_3}$ is changed by $\widetilde{y_3}^{(n)}$. Taking into account monotonic dependence of the minimal solution from the boundary conditions and using the induction by n it is easy to show that $\widetilde{x_k}^{(n)} \leq x_k$, $\widetilde{y_k}^{(n)} \leq y_k$, $k = 1, 2, 3$. From these inequalities follows the convergence of $\left\{\widetilde{x_k}^{(n)}, \widetilde{y_k}^{(n)}\right\}$ to the minimal solution of the problem (0.1)-(0.3). The equality $\Pi_1 \oplus (\Pi_2 \oplus \Pi_3) = \oplus\{\Pi_1, \Pi_2, \Pi_3\}$ is proved. The equality $(\Pi_1 \oplus \Pi_2) \oplus \Pi_3 = \oplus\{\Pi_1, \Pi_2, \Pi_3\}$ may be proved by an analogous manner. Hence the equalities (4.1) take place.

We come to the following theorem:

Theoram 4.1. *The class Ω is a non-commutative semi-group (monoid) with the addition $\oplus$ and zero element θ (see* (4.1)*).*

Corollary (The principle of minimality). *Let $\Pi_1, \cdots, \Pi_N \in \Omega$ and $1 \leq m < N$. Then*

$$\oplus\{\Pi_1, \cdots, \Pi_N\} = \Pi_1 \oplus \cdots \oplus \Pi_N = \{\Pi_1 \oplus \cdots \oplus \Pi_m\} \oplus \{\Pi_{m+1} \oplus \cdots \oplus \Pi_N\} \tag{4.8}$$

(The minimal sum is equal to the minimal sum of the minimal subsumes).

4.2.The solution of the Main problem. The theorem 4.1 gives a method to the solution of the Exit problem, based on the successive construction of sums

$$G_k(\widetilde{\varphi_k}, \widetilde{\psi_k}) = \Pi_1 \oplus \cdots \oplus \Pi_k, \;\; k = 2, \cdots, N.$$

For the construction of the solution of the Main problem we shall use the resolution functions u and v, which will be introduced below.

Let $\Pi_k \in \Omega$, $k = 1, 2$ and x_1, x_2, y_0, y_1 is the minimal solution of the corresponding problem (0.1)-(0.3). The resolution functions u and v are determined by mappings

$$u : (x, y) \to x_1, \;\; v : (x, y) \to y_1. \tag{4.9}$$

Let us describe the procedure of solving of the problem (0.1)-(0.3) using the resolution functions. Proposed scheme, analogous with Gauss method in linear algebra and the method of Invariant Imbedding, consists of direct and inverse runs.

The direct run: a successive construction of sums $G_k(\widetilde{\varphi_k}, \widetilde{\psi_k})$. Each step k of this procedure assumes the construction of the functions $\widetilde{\varphi_k}, \widetilde{\psi_k}$ and the resolution functions u_{k-1}, v_{k-1} for the sum $G_k \oplus \Pi_k$ in accordance with (.), where G_k plays the role of Π_1, and Π_k plays the role of Π_2.

The inverse run. The numbers x_{N-1}, y_{N-1} are determined by formulae

$$x_{N-1} = u_{N-1}(x, y)(= u_{n-1}(x_0, y_N)) \;\; y_{N-1} = v_{N-1}(x, y)(= v_{n-1}(x_0, y_N)).$$

Continue this process, changing N on $N-1$. On k-th step of inverse run, x_{N-k}, y_{N-k} are determined by formulae:

$$x_{N-k} = u_{N-k}(x, y_{N-k+1}), \;\; y_{N-k} = v_{N-k}(x, y_{N-k+1}), \;\; k = 1, 2, \cdots, (N-1). \tag{4.10}$$

The numbers x_N, y_0 are determined by: $x_N = \widetilde{\varphi_N}(x, y), \;\; y_0 = \widetilde{\psi_N}(x, y)$.

Obviously, the numbers x_k, y_k satisfy all of the relations (0.1)-(0.3). Conditions (0.2),(0.3). Using the principe of the minimality (see Corollary to the theorem 4.1) one can show, that these numbers x_k, y_k give the minimal solution of the problem (0.1)-(0.3). The following theorem has been proved:

Theoram 4.2. *Let* $\Pi_k(\varphi_k, \psi_k) \in \Omega, \; k = 1, \cdots, N$ *. Then the Minimal solution of the problem* (0.1)-(0.3) *may be constructed by the recurrent formulas* (4.9).

4.3. Some subsemigroups of Ω. Let us briefly consider some subsemigroups of Ω. All of them have simple physical meaning.

i) The semi-group, consisting of conservative elements (see (1.5)).

ii) The semi-group, consisting of uniformly dissipative elements (see (1.6)).

iii)The semi-group, consisting of elements without sources.

iv) The semi -group, consisting of sums of identical elements (homogeneous semi-group). Each element $\Pi \in \Omega$ generates a semi-group, consisting of $\Pi, \;\Pi \oplus \Pi, \;\Pi \oplus \Pi \oplus \Pi, \cdots$

v) Linear semi-groups with sources and without sources. Then the functions φ, ψ have the form:

$$\begin{aligned} \varphi(x, y) &= q^+ x + r^+ y + \gamma^+, \\ \psi(x, y) &= r^- x + q^- y + \gamma^-, \end{aligned} \tag{4.11}$$

where

$$q^\pm, r^\pm, \gamma^\pm \geq 0, \;\; q^\pm + r^\mp \leq 1, \;\; \gamma^\pm \geq 0. \tag{4.12}$$

The condition of the conservativity means that $q^\pm + r^\mp = 1$. The absence of sources means, that $\gamma^\pm = 0$.

vi) The semi-group with bounded non-linearity. Then the functions φ, ψ have the form:

$$\begin{aligned} \varphi(x, y) &= q^+ x + r^+ y + \gamma^+(x, y), \\ \psi(x, y) &= r^- x + q^- y + \gamma^-(x, y). \end{aligned} \tag{4.13}$$

Here the constants $q^\pm, r^\pm$ satisfying the conditions (4.12), $\gamma^\pm(x, y)$ are bounded non-negative functions, monotonic by x, y and satisfying the condition of continuity from below (see (1.2)).

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