One-sided versus two-sided Gibbs property (continuity)
Two examples
(The role of entropic repulsion)

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Characterising Gibbs measures and
identifying how a measure can fail to be one
is a long-standing interest of me.
Some related questions are being studied here.
Starting from Statistical Mechanics,
I have entered the probabilistic/ergodic area.
Part 1) and 2):
Two examples from Stat. Mech.
Dyson models and Schonmann projection.
One-sided (g) or two-sided (Gibbs) influence decay?
But not both.
If time: Part 3)
Why interest in long-range Dyson models,
with varying decay parameter?
No time Part 4
Half-Lines and Whole Line.
Low-T metastates, and high-T transfer operator eigenfunctions.
Between SRB (Dyn. Syst.)and DLR (Stat. Mech.)

One-sided versus two-sided influence.

Stochastic Systems (Processes).

Shift-invariant measures.

Symbolic Dynamics.

Two flavours:

Time, discrete.

(Dynamical Systems, asymmetric description).

Past and future, one-sided (SRB).

Non-equilibrium, ergodic theory

versus

Space, discrete, here one-dimensional.

(Mathematical Physics, symmetric description).

Left and right, two-sided (DLR).

Equilibrium.

Basic Question 1:
When are both descriptions equivalent?
When not equivalent?
Introduction:
Simple background.
Markov modeling (for the short-sighted..)
Time:
Probability and Statistics (Markov chains).
Future independent of past, given the *present*.
Ergodic Theory, Dynamical Systems.
Ahistoric, forget history.

Space:
Statistical (Mathematical) Physics.
Markov: Inside independent of outside,
given the *border*.
(Take control of your borders).
2-state Markov chains
-timelike-
versus
1-dimensional, nearest neighbour, spin
(e.g. Ising) models
-spacelike-.

Probability measures on e.g.
two-symbol sequences;
configuration space $\Omega = \{-, +\}^{Z}$.
Theorem:
(well-known, see e.g.
Wikipedia lemma "Markov Property",
see further Georgii).
Stationary Markov chains, i.e.
invariant Markov measures on histories,
and n.n. Gibbs measures,
in dimension 1,
are the **same** mathematical objects.
(70s, Brascamp, like me student of Hugenholtz
-recently died, age 101- , and Spitzer,...)
Warning: This is about objects (measures)
on *infinite* time/space.

Question:
If we try to be more far-sighted
does this sameness stay true?
Case 1:
General Ergodic Processes.
Answer: NO!
Gurevich, Ornstein-Weiss, in the 1970's
constructed examples which were
one-sided random, two-sided deterministic.
Thus one-sided entropy (Kolmogorov-Sinai) *positive*
different from two-sided entropy which is *zero*,
and one-sided tail (*trivial*) different
from two-sided tail (*contains everything*).
Remark: Ergodic (invariants) are one-sided objects.
Ergodic Theory is a one-sided theory.
As a non-expert I realised this late.
Well-known, but not widely....

Case 2 :
Restriction to intermediate cases:
Continuous (Quasilocal) Ergodic Processes.
What if we change independent of Markov to
weakly dependent(continuous, quasilocal, almost Markov),
does this sameness between one-sided and two-sided remain true?
Then we exclude the Gurevich and Ornstein-Weiss examples,
which are not continuous (in the product topology).
Various aspects studied by various people.
(Fernández, Gallo, Maillard, Berghout, Verbitskiy, Redig,
Wang, Öberg, Johansson, Pollicott, Walters, den Hollander,Steif,
Chazottes, Sinai, Tempelman...)

Answers:

Sameness with extra regularity conditions
(e.g. Short-range, or Dobrushin Uniqueness): Yes.
One-sided equals two-sided.
(Needed for SRB, Thermodynamic Formalism..).
Without those: NO!
Neither class includes the other.
One direction known since 2011,
(Fernández, Gallo, Maillard).

I will discuss here a more recent (2024) example:
"Schonmann projection",
One-dimensional restriction of pure
two-dimensional n. n. Ising model Gibbs measures at low T.
With Shlosman g-measures.
Non-Gibbs,
slow decay one-sided dependence.
Following Bethuelsen and Conache.

Other direction (2018):
Gibbs measures (non-g) for long-range
Low-T Dyson models (with Bissacot, Endo, Le Ny).

Time version:
Class of Stochastic Processes,
rediscovered repeatedly,
under a variety of names:
(g-measures=
Chains of Infinite Order=
Chains with Complete Connections=
Uniform Martingales/Random Markov
Processes)=
Doeblin measures=
SCUM (Stochastic Chains with Unbounded Memory).
(Keane 70's, Harris 50's,
Onicescu-Mihoc and Doeblin-Fortet 30's,
Kalikow 90's).
Studied in
Ergodic Theory, Probability.

Spatial version:
Gibbs (=DLR) measures=
Gibbs fields=
" almost" Markov random fields.
Discovered independently,
in East (mathematics)
and West (physics),
(Dobrushin, Lanford-Ruelle 60's).
Mathematical Physics.
Here two-state -Bernoulli- variables,
(= **Ising** spins:)
$\omega_i = \pm$, for all $i \in Z$.
(Can be much more general.)
Warning:
DLR Gibbs $\neq$ SRB (Bowen-)Gibbs.

Gibbs measures:

Let G be an infinite graph, here Z.

Configuration space:

Space of sequences: $\Omega = \{-, +\}^G$.

Probability measures on Ω,

labeled by **interactions**.

An interaction is a collection of functions,

$\Phi_X(\omega)$, dependent on $\{-, +\}^X$,

where the X are subsets of G.

Let Λ be a finite subset of G.

We write $\Omega_\Lambda = \{-, +\}^\Lambda$.

Energy (**Hamiltonian**)
$H_\Lambda^{\Phi,\tau}(\omega) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
Sum of interaction-energy terms.
A measure μ is *Gibbs* iff:
(A version of) the
conditional probabilities of
finite-volume configurations,
given the outside configuration, satisfies:
$\mu(\omega_\Lambda | \tau_{\Lambda^c}) = \frac{1}{Z_\Lambda^\tau} \exp - \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
for ALL configurations ω,
boundary conditions τ
and finite volumes Λ.

Gibbsian form.
Rigorous version of
"$\mu = \frac{1}{Z} \exp -H$",
Gibbs canonical ensemble.
Larger energy means
exponentially smaller probability.
Nearest-neighbour interaction means that
$\Phi(X) = 0$,
except when $X = \{i, i+1\}$ or $X = i$,
for some $i \in Z$.
A Gibbs measure for a nearest-neighbour
model satisfies a
spatial Markov property:
$\mu(\omega_{\{1,\ldots,n\}}|\tau_{\{1\ldots,n\}^c}) = \mu(\omega_{\{1\ldots,n\}}|\tau_0\tau_{n+1})$.
Conditioned on the border spins,
at 0 and $n+1$,
inside and *outside* are independent.

A two-state Markov chain is again
a measure on the same sequence space Ω.
Now it has to satisfy the " ordinary"
(timelike) Markov property:
$\mu(\omega_{\{1\ldots n\}}|\tau_{\{-\infty,\ldots,0\}}) = \mu(\omega_{\{1\ldots n\}}|\tau_0)$.
One can describe this via a product
of 2-by-2 stochastic matrices P
with non-zero entries:
$P(k,l) = P(\omega_i = k \rightarrow \omega_{i+1} = l)$.
Here $k, l = \pm$ and i is any site (=time) in Z.
There is a one-to-one connection between
stationary (time-invariant)
2-state Markov Chains
and (space-translation-invariant) nearest-neighbor
Ising Gibbs measures.

Continuity (=almost Markov = quasilocality).
Product topology:
Two sequences are close if they are
equal on a large enough finite interval.
Topology metrisable,
metric e.g. by:
$d(\omega, \omega') = 2^{-|n|}$,
where n is the site with
minimal distance from origin such that
$\omega_n \neq \omega'_n$.
A function is continuous,
if it depends weakly on sites far away
and mostly on what happens not too far,
(or not too long ago)
whatever it is.

Processes (time):
$\mu(\sigma_0 = \omega_0 | \omega_{Z-}) = g(\omega_0 \omega_{Z-})$,
with g-function continuous.
Probability of getting ω_0, given the past.
Continuous dependence on the **past**.
Continuity studied since the 30's
(Doeblin-Fortet).
Claim!?:
Continuity implies uniqueness (Harris(50's)).
Mistake in proof pointed out by Keane (70's).
Counterexamples due to Bramson-Kalikow (90's).
Sharper criterion Berger-Hoffman-Sidoravicius (2003-2017).

Gibbs measures:

Continuity of two-sided conditional probabilities
(more or less) corresponds to summability of interactions.
$\sum_{0 \in X} ||\Phi_X|| < \infty$.
Continuous dependence on **outside**
beyond the border.
(Quasilocality).
No action at a distance.
(No observable influence from behind the moon)
Plus: "non-nullness".
Any finite change in the -infinite- system
costs a finite amount of energy.
Any configuration in finite domain
occurs with finite probability,
whatever is happening outside.
Gibbs measures satisfy (equivalently) a
finite-energy condition.

Equivalence holds (Kozlov-Sullivan; Barbieri et al):
Finite-energy + continuity = Gibbs.
Various measures can be shown to be *non-Gibbs*.
This has been for me a major research topic.
Here related questions with weakened "non-nullness" were studied.

Short SRB detour:

Gibbs according to SRB (Bowen-Gibbs) is for $d = 1$.

More general than DLR Gibbs

because configuration space typically

subshift instead of full shift.

More options.

Less general because more restrictive

condition on interactions.

$\sum_{0 \in X} ||\Phi_X|| diam(X) < \infty$.

First-moment condition, stronger than summability.

Note that in $d = 1$ $diam(X) > |X|$,

the diameter of X is larger than the number of sites in X.

Now back to continuity and summability.

For g-measure,
$var_n\ g(\omega) = \sup_{\eta,\sigma}[g(\omega_n\eta) - g(\omega_n\sigma)]$.
$var_n\ g = \sup_\omega var_n g(\omega)$.
Continuity means $\lim_{n\to\infty} var_n = 0$.
Summability of $var_n\ g$
implies uniqueness AND equivalence
between g and Gibbs.
Short-range condition (all temperatures).
Other criterion (high-temperature, or strong field)
"Dobrushin uniqueness" also sufficient,
both for uniqueness AND equivalence.

Our Dyson Counterexample:
(Gibbs, non-g-measure).
Gibbs measures for Dyson models.
Low temperatures.
Long-range Ising models.
Ferromagnetic pair interactions.
$\Phi_{i,j}(\omega) = -J|i-j|^{-\alpha}\omega_i\omega_j$.
Interesting regime $1 < \alpha \leq 2$.
Phase transition for large J,
at low temperatures:
There exist then **two** different
Gibbs measures, for the same interaction,
called μ^+ and μ^-, for such Φ.
Microscopic interfaces **don't exist**.
(One-dimensional Aizenman-Higuchi)
Spatially continuous conditional probabilities.
Warning:
Impossible for Markov Chains or Fields,
always uniqueness.

Claim:
At low T and for $\alpha^* < \alpha < 2$
Dyson Gibbs measures are not g-measures.
(At high T they are by Dobrushin uniqueness).
Here technical condition $\alpha^* = 3 - \frac{\ln 3}{\ln 2}$.
(Removable by Corsini thesis (?))
Proof uses technically rather hard,
largely Italian, Input,
perturbative, cluster expansions, from others,
giving the α^* condition,
plus three simple Observations.

Input:

Interface result for Dyson models
(Cassandro, Merola, Picco, Rozikov).
Take interval $[-L, +L]$,
all spins to the left are minus,
all spins to the right are plus.
Then there is an interface point **IF**P, such that:
1) To the left of the interface
we are in the minus phase (μ^-),
to the right of the interface
we are in the plus phase (μ^+).
2) With overwhelming probability the location
of the interface is at most $O(L^{\frac{\alpha}{2}})$ from the center.
$\ldots - - - - - m \ldots . |\mathbf{IFP}| + m \ldots .. | + + + + + \ldots$

Observation 1:
If I change all spins to the left of a length-N interval of minuses,
the effect from the left
on the central O(L) interval
is bounded by $O(LN^{1-\alpha})$,
thus small for N large.

Consequence:
A large interval of minuses (size N)
will have a moderately large (size L)
interval of minus phase on both sides.
Interfaces are pushed away.
Entropic repulsion from fluctuations, mesoscopic.
Entropic repulsion on the line (longitudinal).

Observation 2:
If I decouple a comparatively small interval,
of size $L_1 = o(L)$,
inside the beginning (left side)
of my minus-phase interval,
this hardly changes the interface location.
(Cost of **IFP** shift by εL is larger, namely $O(L^{2-\alpha})$.
Shown by Cassandro et al.)

Observation 3:
If I make in this L_1 interval
an alternating configuration
+-+-+-+-+-...
then the total energy (influence)
on its complement
is bounded by the double sum
$\sum_{i=1....L_1,j>L_1}(|j-i|^{-\alpha}-|j+1-i|^{-\alpha})=$
$\sum_{i=1....L_1,j>L_1}(O(|j-i|^{-(\alpha+1)})=$
$\sum_{i=1....L_1}O(|i|^{-\alpha})$
which is bounded, uniformly in L_1.
Therefore finite, small effect.

Remark:

Effect only at positive non-zero temperature.

Entropic Repulsion.

No entropy at $T = 0$.

A large alternating interval,
preceded by a MUCH
larger interval of minuses,
cannot shield the influence
of this homogeneous minus interval.

But this means precisely that
the conditional probability of finding
a plus (or a minus),
at a given site,
conditioned on an alternating past,
is not continuous.
Thus two-sided continuity
occurring at the same time
as one-sided discontinuity.

Alternating configuration is discontinuity point,
due to cancellations of pluses and minuses.
Set of discontinuity points may have measure zero,
but nonremovable.
$... - - - - - + - + - + - X$ (- N, alt_L intervals)
versus
$... + + + + - + - + - + - X$ (+N, alt_L intervals)
Expected value of X differs,
by more than cst,
uniformly in L and N(L).
Direct influence from Deep Past.

QUESTION:
ALL $1 < \alpha < 2$?
Sao Paulo group (Bissacot,Corsini,Maia, Welsch...)
Some results are there. Generalise this entropic repulsion and non-g result?

Example 2

Schonmann projection.

1d projection of 2d low-T Gibbs measure of n.n. Ising model, μ^+.

Non-Gibbsian.

(Schonmann, later analyses by v.E-Fernández-Sokal, Fernández-Pfister, Dobrushin-Shlosman, Bricmont-Kupiainen-Lefevere, Maes-VandeVelde, Maes-Redig-van Moffaert, Fernández-Le Ny-Redig-)

Main reason: transversal entropic repulsion.

Wetting.

Condition on large interval of minus spins, of length N.
Then around this interval is a
large (size $2N + O(\sqrt{N})$) contour.
Contour is at transversal distance $O(\sqrt{N})$ of the interval.
Both above and below. (Cigar-shaped.)
-Entropic repulsion-.
Now take two alternating intervals around the center C, both of size n.
$n << N$.

$+$m- - - - - - - $+$-$+$-$+$-$+$-.$+$-$+$-$+$-$+$- - - - - - $+$m

N, n, C, n, N,

The alternating intervals provide an
energetic attraction proportional to n.
Entropic repulsion grows to infinity with N.
It will dominate when N is sufficiently large wrt n.
But if the Center of the interval is inside a contour,
its preference is to be minus.
So the two HUGE minus intervals,
at large distance n from the Center
influence the probability of the Center spin to be minus,
with strength uniformly in n.
No continuity: Non-Gibbs.

Claim 2: The Schonmann projection μ^+ is a $g-$measure for a continuous function g^+.

1) For μ^+ almost all ω $g_n^+(\omega)$ decays exponentially fast in n. (Bethuelsen-Conache).

Almost all, but not all!

As μ^+ is non-Gibbsian,

there must be configurations for which this is not true.

(Summability would imply Gibbs).

2) The worst kind of behaviour occurs for

$\omega = -1$, the all-minus configuration.

Very atypical for μ^+.

Proof idea:
Contour around minus interval I_N
behaves like two N-step random walk excursions,
one completely below, the other completely above the interval.
The fraction of excursions
(from all walks) behaves like
$O(N^{-\frac{3}{2}})$.
Square that (two excursions) $O(N^{-3})$.
(Ioffe-Ott-Velenik-Wachtel).
The probability of adding a minus to the right of the interval
= ratio of 2 partition functions.
Compare number of contours around I_{N+1} and I_N,
$Cst \frac{(N+1)^{-3}}{N^{-3}} = O(\frac{1}{N})$.
Energies cancel, and give Cst.
Only entropy plays an N-dependent role,
but it creates a "long-range" g-function.

$g^+(- - - -_N \omega_{N^c})$ converges to zero,
uniform in ω (continuous), but slowly.
Still square summable, so uniqueness holds.
(Johannson, Öberg, Pollicott).
Plus and Minus Schonmann projections
are g-measures for different g-functions.

Remarks:

Decimation

(marginal on periodic subsequence $kZ, K > 1$).

Opposite results:

1) Decimating the low-T Dyson Gibbs measures produces non-Gibbsian measures.

(AvE with Le Ny (and d'Achille).

2) Decimating the non-Gibbsian Schonmann projections produces Gibbs measures.

(Lörinczi and VandeVelde).

Remarks on the long-range parameter α
of Dyson models.
Correspondences (imperfect)
Simulate
short-range models in higher dimensions.
Continuously varying dimensions, well-defined.
Renormalisation Group (RG).

1)
At low T surface versus volume analogy
$2 - \alpha = \frac{d-1}{d}$, thus
$\alpha = 1 + \frac{1}{d}$.
Transition thresholds $d = 1$, (Campari-Cassi).
$\alpha = 2$. (Complicated).
2)
At the critical temperature one has:
$\alpha = 1 + \frac{2}{d}$.
Mean-field behaviour threshold
$d = 4$, corresponds to $\alpha = \frac{3}{2}$. (RG, ε- expansions)
3)
SOS and discrete Gaussian (infinite spins) in $d = 1$.
Localised $1 < \alpha < 2$. (Coquille,Dario,Le Ny).
Delocalised $\alpha > 2$.
Normal fluctuations $\alpha > 3$.
Interfaces for 2d long-range models.

Half-lines versus whole lines.
Zero field.
Consider the interaction between two half-lines.
$W = \sum_{i<0,j>0} |i-j|^{-\alpha}\eta_i\sigma_j$.

Then μW and also $\mu \exp W$
are finite at high T,
between two random half-lines,
(at high T the correlation decay as the interaction
$\mu(\eta_i\sigma_j) = O(|i-j|^{-\alpha})$,
$(\sum_{i<0,j>0} |i-j|^{-2\alpha} < \infty$.
but infinite at low T, (between two ordered half- lines.)
(That is why a phase transition can take place).
$\sum_{i<0,j>0} |i-j|^{-\alpha} = \infty$.

New observation:

Consider a random half-line on the left,
and an ordered (plus) half-line on the right.
Then the interaction energy W between them is
finite for $\alpha > \frac{3}{2}$,
infinite otherwise:
$\sum_{-N,..-1} \eta_i |i|^{1-\alpha} = O(N^{\frac{3}{2}-\alpha})$.
For i.i.d. central limit theorem for
nonidentical independent variables,
for high T "random" means "weakly dependent",
same behaviour as independent.

Metastates with random b.c. at low T.
Limit Gibbs states are either
randomly mixed states $\frac{3}{2} < \alpha$
or pure states (plus or minus)
with some probability. (v.E., Le Ny, Endo).
Transfer matrix eigenfunctions at high T
half-line RN densities, continuous or just L^1.
Johannson, Öberg,Pollicott
vE, Fernandéz. Makhmudov, Verbitskiy.
Special values: $\alpha = 1,\ \frac{3}{2},\ 2,\ 3.$.

Conclusions:

a) Two-sided continuous dependence
-spacelike- does *not* imply
one-sided continuous dependence
-timelike.
But neither is it implied.

Summary:

Controlling borders is NOT the same as
control of history,
except for the shortsighted.

b)Every counterexample needs to be somewhat long-range.
Either interaction,or g-function.

c)High-T continuity of Dyson eigenfunctions of transfer matrices,
Low-T metastates living on mixed states,
have the same origin.

Two open questions:
Dyson model.
Question:
Understand $\alpha = 2$ case
(open, no mesoscopic localisation of interface point).
Entropic Repulsion?

Schonmann projection.
Question:
If I condition μ^+ on
interval (line) of minus magnetisation,
is there transversal entropic repulsion?
True for all minus, but what about "typical" minus?
"How bad" non-Gibbsianness?

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